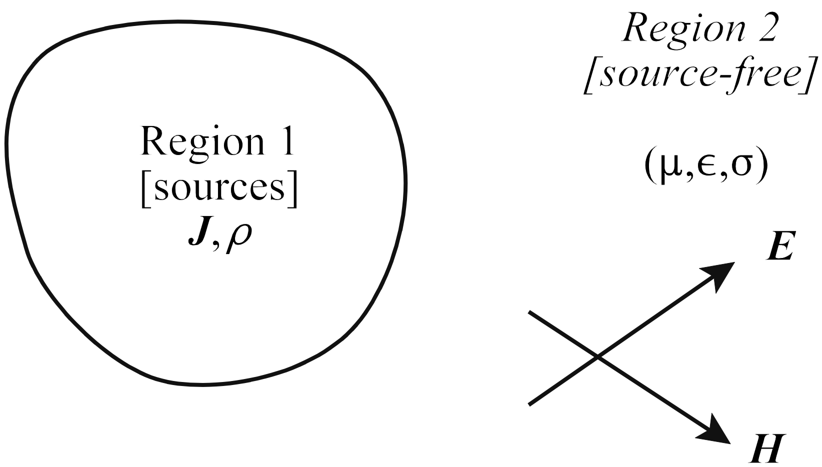
Electromagnetic Waves

Maxwell's equations predict the propagation of electromagnetic energy away from time-varying sources (current and charge) in the form of waves. Consider a linear, homogeneous, isotropic media characterized by (μ, ϵ, σ) in a source-free region (sources in region 1, source-free region is region 2).



We start with the source-free, instantaneous Maxwell's equations written in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ only. Note that conduction current in the source-free region is accounted for in the $\sigma\boldsymbol{E}$ term.

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \qquad ①$$

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t} \qquad ②$$

$$\nabla \cdot \boldsymbol{E} = 0 \qquad ③$$

$$\nabla \cdot \boldsymbol{H} = 0 \qquad ④$$

Taking the curl of ①

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \boldsymbol{H})$$

and inserting ② gives

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \boldsymbol{E} + \epsilon \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$= -\mu \sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \text{⑤}$$

Taking the curl of ②

$$\nabla \times \nabla \times \boldsymbol{H} = \sigma(\nabla \times \boldsymbol{E}) + \epsilon \frac{\partial}{\partial t}(\nabla \times \boldsymbol{E})$$

and inserting ① yields

$$\nabla \times \nabla \times \boldsymbol{H} = \sigma\left(-\mu \frac{\partial \boldsymbol{H}}{\partial t}\right) + \epsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \boldsymbol{H}}{\partial t}\right)$$

$$= -\mu \sigma \frac{\partial \boldsymbol{H}}{\partial t} - \mu \epsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2} \qquad \text{⑥}$$

Using the vector identity

$$\nabla \times \nabla \times \boldsymbol{F} = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F} \qquad (\text{for any vector } \boldsymbol{F})$$

in ⑤ and ⑥ gives

$$\nabla \times \nabla \times \boldsymbol{E} = \nabla(\overbrace{\nabla \cdot \boldsymbol{E}}^{0\ (\text{from ③})}) - \nabla^2 \boldsymbol{E} = -\mu \sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu \epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla \times \nabla \times \boldsymbol{H} = \nabla(\overbrace{\nabla \cdot \boldsymbol{H}}^{0\ (\text{from ④})}) - \nabla^2 \boldsymbol{H} = -\mu \sigma \frac{\partial \boldsymbol{H}}{\partial t} - \mu \epsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{E} = \mu \sigma \frac{\partial \boldsymbol{E}}{\partial t} + \mu \epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{H} = \mu \sigma \frac{\partial \boldsymbol{H}}{\partial t} + \mu \epsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

Instantaneous
vector wave equations
(Helmholtz equations)

For time-harmonic fields, the instantaneous (time-domain) vector $\boldsymbol{F}$ is related to the phasor (frequency-domain) vector $\boldsymbol{F}_s$ by

$$\boldsymbol{F} \quad \Leftrightarrow \quad \boldsymbol{F}_s$$

$$\frac{\partial \boldsymbol{F}}{\partial t} \quad \Leftrightarrow \quad j\omega \boldsymbol{F}_s$$

$$\frac{\partial^2 \boldsymbol{F}}{\partial t^2} \quad \Leftrightarrow \quad (j\omega)^2 \boldsymbol{F}_s$$

Using these relationships, the instantaneous vector wave equations are transformed into the phasor vector wave equations:

$$\nabla^2 \boldsymbol{E}_s = \mu\sigma(j\omega)\boldsymbol{E}_s + \mu\epsilon(j\omega)^2\boldsymbol{E}_s = j\omega\mu(\sigma + j\omega\epsilon)\boldsymbol{E}_s$$

$$\nabla^2 \boldsymbol{H}_s = \mu\sigma(j\omega)\boldsymbol{H}_s + \mu\epsilon(j\omega)^2\boldsymbol{H}_s = j\omega\mu(\sigma + j\omega\epsilon)\boldsymbol{H}_s$$

If we let

$$j\omega\mu(\sigma + j\omega\epsilon) = \gamma^2$$

the phasor vector wave equations reduce to

$$\nabla^2 \boldsymbol{E}_s - \gamma^2 \boldsymbol{E}_s = 0$$

$$\nabla^2 \boldsymbol{H}_s - \gamma^2 \boldsymbol{H}_s = 0$$

Phasor vector
wave equations
(Helmholtz equations)

The complex constant γ is defined as the *propagation constant*.

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta$$

The real part of the propagation constant (α) is defined as the *attenuation constant* while the imaginary part (β) is defined as the *phase constant*. The attenuation constant defines the rate at which the fields of the wave are attenuated as the wave propagates. An electromagnetic wave propagates in an ideal (lossless) media without attenuation ($\alpha = 0$). The phase constant defines the rate at which the phase changes as the wave propagates.

Separate but equivalent units are defined for the propagation, attenuation and phase constants in order to identify each quantity by its units [similar to complex power, with units of VA (complex power), W (real power) and VAR (reactive power)].

γ	propagation constant (m^{-1})
α	attenuation constant (Np/m)
β	phase constant (rad/m)

Given the properties of the medium (μ, ϵ, σ), we may determine equations for the attenuation and phase constants.

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = (\alpha + j\beta)^2 = \alpha^2 + j2\alpha\beta - \beta^2$$

$$\left.\begin{aligned} Re\,\gamma^2 &= \alpha^2 - \beta^2 = -\omega^2\mu\epsilon \\ Im\,\gamma^2 &= 2\alpha\beta = \omega\mu\sigma \end{aligned}\right\} \text{Solve for } \alpha,\beta$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]}$$

Properties of Electromagnetic Waves

The properties of an electromagnetic wave (direction of propagation, velocity of propagation, wavelength, frequency, attenuation, etc.) can be determined by examining the solutions to the wave equations that define the electric and magnetic fields of the wave. In a source-free region, the phasor vector wave equations are

$$\nabla^2 \boldsymbol{E}_s = \gamma^2 \boldsymbol{E}_s$$

$$\nabla^2 \boldsymbol{H}_s = \gamma^2 \boldsymbol{H}_s$$

The operator in the above equations (∇^2) is the *vector Laplacian* operator. In rectangular coordinates, the vector Laplacian operator is related to the scalar Laplacian operator as shown below.

$$\boldsymbol{F}_s = F_{xs}\boldsymbol{a}_x + F_{ys}\boldsymbol{a}_y + F_{zs}\boldsymbol{a}_z$$

$$\nabla^2 \boldsymbol{F}_s = (\nabla^2 F_{xs})\boldsymbol{a}_x + (\nabla^2 F_{ys})\boldsymbol{a}_y + (\nabla^2 F_{zs})\boldsymbol{a}_z$$

Vector Laplacian

Scalar Laplacian

$$\nabla^2 f_s = \frac{\partial^2 f_s}{\partial x^2} + \frac{\partial^2 f_s}{\partial y^2} + \frac{\partial^2 f_s}{\partial z^2}$$

The phasor wave equations can then be written as

$$(\nabla^2 E_{xs})\boldsymbol{a}_x + (\nabla^2 E_{ys})\boldsymbol{a}_y + (\nabla^2 E_{zs})\boldsymbol{a}_z = \gamma^2 (E_{xs}\boldsymbol{a}_x + E_{ys}\boldsymbol{a}_y + E_{zs}\boldsymbol{a}_z)$$

$$(\nabla^2 H_{xs})\boldsymbol{a}_x + (\nabla^2 H_{ys})\boldsymbol{a}_y + (\nabla^2 H_{zs})\boldsymbol{a}_z = \gamma^2 (H_{xs}\boldsymbol{a}_x + H_{ys}\boldsymbol{a}_y + H_{zs}\boldsymbol{a}_z)$$

Individual wave equations for the phasor field components [(E_{xs}, E_{ys}, E_{zs}) and (H_{xs}, H_{ys}, H_{zs})]can be obtained by equating the vector components on both sides of each phasor wave equation.

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = \gamma^2 E_{xs}$$

$$\frac{\partial^2 E_{ys}}{\partial x^2} + \frac{\partial^2 E_{ys}}{\partial y^2} + \frac{\partial^2 E_{ys}}{\partial z^2} = \gamma^2 E_{ys}$$

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} = \gamma^2 E_{zs}$$

$$\frac{\partial^2 H_{xs}}{\partial x^2} + \frac{\partial^2 H_{xs}}{\partial y^2} + \frac{\partial^2 H_{xs}}{\partial z^2} = \gamma^2 H_{xs}$$

$$\frac{\partial^2 H_{ys}}{\partial x^2} + \frac{\partial^2 H_{ys}}{\partial y^2} + \frac{\partial^2 H_{ys}}{\partial z^2} = \gamma^2 H_{ys}$$

$$\frac{\partial^2 H_{zs}}{\partial x^2} + \frac{\partial^2 H_{zs}}{\partial y^2} + \frac{\partial^2 H_{zs}}{\partial z^2} = \gamma^2 H_{zs}$$

The component fields of any time-harmonic electromagnetic wave (described in rectangular coordinates) must individually satisfy these six partial differential equations. In many cases, the electromagnetic wave will not contain all six components. An example of this is the *plane wave*.

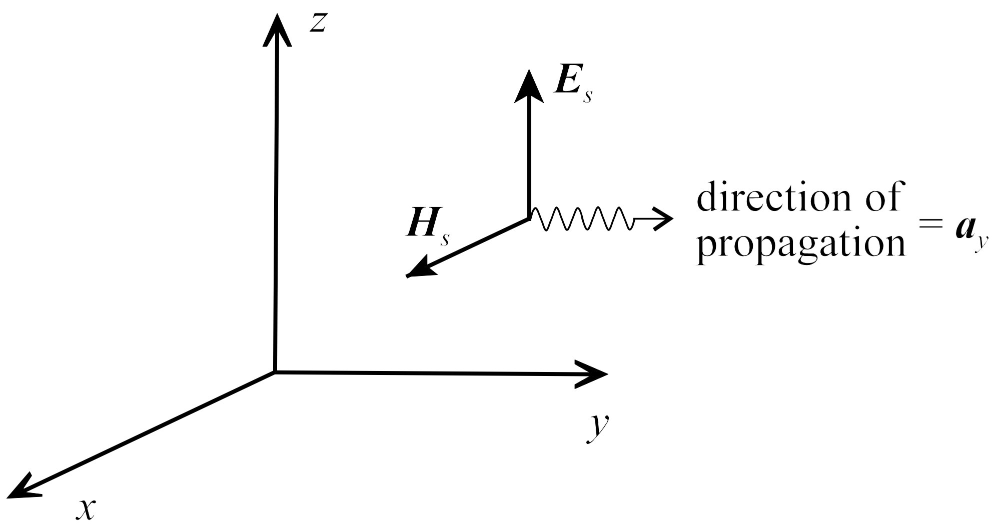
Plane Wave

(1.) ***E*** and ***H*** lie in a plane ⊥ to the direction of propagation.
(2.) ***E*** and ***H*** are ⊥ to each other.

Uniform Plane Wave

In addition to (1.) and (2.) above, ***E*** and ***H*** are uniform in the plane ⊥ to the direction of propagation (***E*** and ***H*** vary only in the direction of propagation).

Example (Uniform time-harmonic plane wave)



$$\boldsymbol{E_s} = E_{zs}(y)\,\boldsymbol{a_z}$$

$$\boldsymbol{H_s} = H_{xs}(y)\,\boldsymbol{a_x}$$

The uniform plane wave for this example has only a z-component of electric field and an x-component of magnetic field which are both functions of only y. An electromagnetic wave which has no electric or magnetic field components in the direction of propagation (all components of $\boldsymbol{E}$ and $\boldsymbol{H}$ are perpendicular to the direction of propagation) is called a *transverse electromagnetic (TEM) wave*. All plane waves are TEM waves. The *polarization* of a plane wave is defined as the direction of the electric field (this example is a z-polarized plane wave). For this uniform plane wave, the component wave equations for the only two field components (E_{zs}, H_{xs}) can be simplified significantly given the field dependence on y only.

$$\cancelto{0}{\frac{\partial^2 E_{zs}}{\partial x^2}} + \frac{\partial^2 E_{zs}}{\partial y^2} + \cancelto{0}{\frac{\partial^2 E_{zs}}{\partial z^2}} = \gamma^2 E_{zs}$$

$$\cancelto{0}{\frac{\partial^2 H_{xs}}{\partial x^2}} + \frac{\partial^2 H_{xs}}{\partial y^2} + \cancelto{0}{\frac{\partial^2 H_{xs}}{\partial z^2}} = \gamma^2 H_{xs}$$

The remaining single partial derivative in each component wave equation becomes a pure derivative since E_{zs} and H_{xs} are functions of y only.

$$\frac{d^2 E_{zs}}{dy^2} - \gamma^2 E_{zs} = 0$$

$$\frac{d^2 H_{xs}}{dy^2} - \gamma^2 H_{xs} = 0$$

Linear, homogeneous, second order D.E.'s

The general solutions to the reduced waves equations are

$$\begin{aligned} E_{zs}(y) &= E_1 e^{\gamma y} + E_2 e^{-\gamma y} \\ &= E_1 e^{(\alpha + j\beta)y} + E_2 e^{-(\alpha + j\beta)y} \\ &= E_1 e^{\alpha y} e^{j\beta y} + E_2 e^{-\alpha y} e^{-j\beta y} \end{aligned}$$

$$\begin{aligned} H_{xs}(y) &= H_1 e^{\gamma y} + H_2 e^{-\gamma y} \\ &= H_1 e^{(\alpha + j\beta)y} + H_2 e^{-(\alpha + j\beta)y} \\ &= H_1 e^{\alpha y} e^{j\beta y} + H_2 e^{-\alpha y} e^{-j\beta y} \end{aligned}$$

where (E_1, E_2) are constants (electric field amplitudes) and (H_1, H_2) are constants (magnetic field amplitudes). Note that E_{zs} and H_{xs} satisfy the same differential equation. Thus, other than the field amplitudes, the wave characteristics of the fields are identical.

The characteristics of the waves defined by the general field solutions above can be determined by investigating the corresponding instantaneous fields. We may focus on either the electric field or the magnetic field since they both have the same wave characteristics (they both satisfy the same differential equation).

$$\begin{aligned} E_z(y,t) &= Re\left\{E_{zs}(y)\, e^{j\omega t}\right\} \\ &= Re\left\{\left(E_1 e^{\alpha y} e^{j\beta y} + E_2 e^{-\alpha y} e^{-j\beta y}\right) e^{j\omega t}\right\} \\ &= Re\left\{ E_1 e^{\alpha y} e^{j(\omega t + \beta y)} + E_2 e^{-\alpha y} e^{j(\omega t - \beta y)} \right\} \\ &= E_1 e^{\alpha y} \cos(\omega t + \beta y) + E_2 e^{-\alpha y} \cos(\omega t - \beta y) \end{aligned}$$

$$E_{zs}(y,\omega) = E_1 e^{\alpha y} e^{j\beta y} + E_2 e^{-\alpha y} e^{-j\beta y} \qquad \begin{pmatrix}\text{frequency}\\ \text{domain}\end{pmatrix}$$

$$E_z(y,t) = \underbrace{E_1 e^{\alpha y}\cos(\omega t + \beta y)} + \underbrace{E_2 e^{-\alpha y}\cos(\omega t - \beta y)} \qquad \begin{pmatrix}\text{time}\\ \text{domain}\end{pmatrix}$$

Amplitude = $E_1 e^{\alpha y}$	Amplitude = $E_2 e^{-\alpha y}$
Phase = $\omega t + \beta y$	Phase = $\omega t - \beta y$
grows in $+\boldsymbol{a}_y$ direction	decays in $+\boldsymbol{a}_y$ direction
decays in $-\boldsymbol{a}_y$ direction	grows in $-\boldsymbol{a}_y$ direction
($-\boldsymbol{a}_y$ traveling wave)	($+\boldsymbol{a}_y$ traveling wave)
$\omega t + \beta y$ = constant	$\omega t - \beta y$ = constant
βy = constant $- \omega t$	$\beta y = \omega t$ $-$ constant
as $t \uparrow, y \downarrow$	as $t \uparrow, y \uparrow$
($-\boldsymbol{a}_y$ traveling wave)	($+\boldsymbol{a}_y$ traveling wave)

$\Leftarrow$ To locate a point of constant phase on the wave

The velocity at which this point of constant phase moves is the *velocity of propagation* for the wave. Solving for the position variable y in the equations defining the point of constant phase gives

$$y = \pm\frac{1}{\beta}(\omega t - \text{constant}) \qquad (\pm\,\boldsymbol{a}_y \text{ traveling wave})$$

Given the y-coordinate of the constant phase point as a function of time, the vector velocity $\boldsymbol{u}$ at which the constant phase point moves is found by differentiating the position function with respect to time.

$$\boldsymbol{u} = \frac{dy}{dt}\boldsymbol{a}_y = \pm\frac{\omega}{\beta}\boldsymbol{a}_y = \frac{\omega}{\beta}(\pm\boldsymbol{a}_y) \qquad (\pm\,\boldsymbol{a}_y \text{ traveling wave})$$

$$u = \frac{\omega}{\beta} \qquad \left(\text{velocity of propagation}, \ \frac{\text{m}}{\text{s}}\right)$$

$$\omega = 2\pi f = \frac{2\pi}{T} \qquad \left(\text{radian frequency}, \ \frac{\text{rad}}{\text{s}}\right)$$

Given a wave traveling at a velocity u, the wave travels one wavelength (λ) during one period (T).

$$\lambda = uT = \frac{u}{f} = \frac{\omega/\beta}{f} = \frac{2\pi}{\beta} \qquad (\text{wavelength , m})$$

$$\beta = \frac{2\pi}{\lambda} \qquad \left(\text{phase change} = \frac{2\pi \text{ radians}}{\text{wavelength}}\right)$$

For a uniform plane wave propagating in a given medium, the ratio of electric field to magnetic field is a constant. The units on this ratio has units of ohms and is defined as the *intrinisic wave impedance* for the medium. Assuming a $+\boldsymbol{a}_y$ traveling uniform plane wave defined by an electric field of

$$\boldsymbol{E}_s = E_{zs}\boldsymbol{a}_z = E_o e^{-\gamma y}\boldsymbol{a}_z$$

the corresponding magnetic field is found from the source free Maxwell's equations.

$$\nabla \times \boldsymbol{E}_s = -j\omega\mu\boldsymbol{H}_s$$

$$\begin{aligned}
\boldsymbol{H}_s = -\frac{1}{j\omega\mu}\nabla \times \boldsymbol{E}_s &= -\frac{1}{j\omega\mu}\left[\frac{\partial E_{zs}}{\partial y}\boldsymbol{a}_x - \frac{\partial E_{zs}}{\partial x}\boldsymbol{a}_y\right] \\
&= -\frac{1}{j\omega\mu}\left[\frac{\partial}{\partial y}(E_o e^{-\gamma y})\,\boldsymbol{a}_x\right] \\
&= -\frac{1}{j\omega\mu}(-\gamma E_o e^{-\gamma y})\,\boldsymbol{a}_x \\
&= \frac{\gamma}{j\omega\mu}E_o e^{-\gamma y}\boldsymbol{a}_x \\
&= H_{xs}\boldsymbol{a}_x
\end{aligned}$$

Note that the direction of propagation for this wave is in the same direction as $\boldsymbol{E}\times\boldsymbol{H}$ ($\boldsymbol{a}_z\times\boldsymbol{a}_x = \boldsymbol{a}_y$). This characteristic is true for all plane waves.

The intrinsic impedance of the wave is defined as the ratio of the electric field and magnetic field phasors (complex amplitudes).

$$\eta = \frac{E_{zs}}{H_{xs}} = \frac{E_o e^{-\gamma y}}{\dfrac{\gamma}{j\omega\mu} E_o e^{-\gamma y}} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\epsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

$$\eta = |\eta|\, e^{j\theta_\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \qquad (\text{intrinsic wave impedance}, \ \Omega)$$

In general, the intrinsic wave impedance is complex. The magnitude of the complex intrinsic wave impedance is

$$|\eta| = \frac{\sqrt{\dfrac{\mu}{\epsilon}}}{\left[1 + \left(\dfrac{\sigma}{\omega\epsilon}\right)^2\right]^{-1/4}}$$

Summary of Wave Characteristics − Lossy Media (General case)

Lossy media ⇒ $(\sigma > 0,\ \mu = \mu_r \mu_o,\ \epsilon = \epsilon_o \epsilon_r)$

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta \qquad \text{(complex)}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1\right]} \qquad \beta = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1\right]}$$

$$u = \frac{\omega}{\beta} \qquad \lambda = \frac{2\pi}{\beta} \qquad \eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \qquad \text{(complex)}$$

Summary of Wave Characteristics − Lossless Media

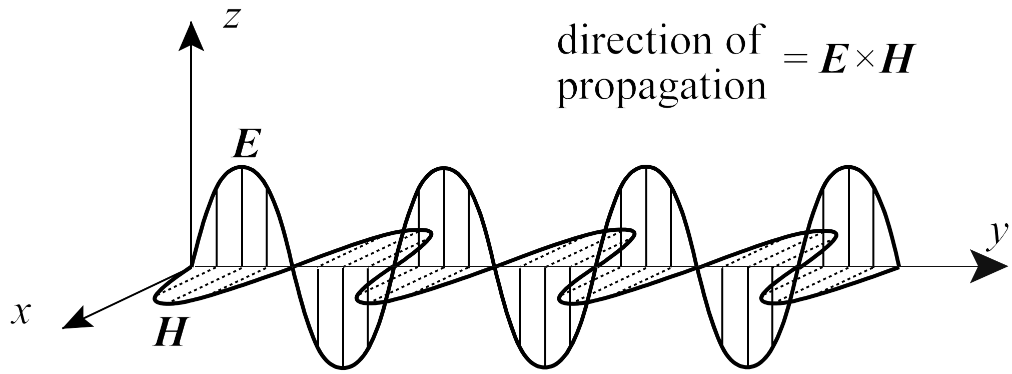
Lossless media ⇒ $(\sigma = 0,\ \mu = \mu_r \mu_o,\ \epsilon = \epsilon_o \epsilon_r)$

$$\gamma = \sqrt{-\omega^2\mu\epsilon} = j\omega\sqrt{\mu\epsilon} = \alpha + j\beta \qquad \text{(imaginary)}$$

$$\alpha = 0 \qquad \beta = \omega\sqrt{\mu\epsilon} = \omega\sqrt{\mu_r\mu_o\epsilon_r\epsilon_o} = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r}$$

$$u = \frac{\omega}{\beta} = \frac{c}{\sqrt{\mu_r\epsilon_r}} \qquad \lambda = \frac{2\pi}{\beta} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}} \qquad \text{(real)}$$

The figure below shows the relationship between ***E*** and ***H*** for the previously assumed uniform plane wave propagating in a lossless medium. The lossless medium propagation constant is purely imaginary ($\gamma = j\beta$) while the intrinsic wave impedance is purely real.



$$\boldsymbol{E}_s = E_o e^{-j\beta y} \boldsymbol{a}_z \qquad \boldsymbol{H}_s = \frac{E_o}{\eta} e^{-j\beta y} \boldsymbol{a}_x$$

$$\boldsymbol{E} = E_o \cos(\omega t - \beta y) \boldsymbol{a}_z \qquad \boldsymbol{H} = \frac{E_o}{\eta} \cos(\omega t - \beta y) \boldsymbol{a}_x$$

Note that the electric field and magnetic field in a lossless medium are in phase.

For a lossy medium, the only difference in the figure above would be an exponential decay in both ***E*** and ***H*** in the direction of wave propagation. The propagation constant and the intrinsic wave impedance of a lossy medium are complex ($\gamma = \alpha + j\beta$, $\eta = |\eta| e^{j\theta_\eta}$) which yields the following electric field and magnetic fields:

$$\boldsymbol{E}_s = E_o e^{-\alpha y} e^{-j\beta y} \boldsymbol{a}_z \qquad \boldsymbol{H}_s = \frac{E_o}{\eta} e^{-\alpha y} e^{-j\beta y} \boldsymbol{a}_x$$

$$\boldsymbol{E} = E_o e^{-\alpha y} \cos(\omega t - \beta y) \boldsymbol{a}_z \qquad \boldsymbol{H} = \frac{E_o}{|\eta|} e^{-\alpha y} \cos(\omega t - \beta y - \theta_\eta) \boldsymbol{a}_x$$

The electric and magnetic fields in a lossy medium are out of phase an amount equal to the phase angle of the intrinsic impedance.

Wave Propagation in Free Space

Air is typically very low loss (negligible attenuation) with little polarization or magnetization. Thus, we may model air as free space (vacuum) with $\sigma=0$, $\epsilon=\epsilon_o$, and $\mu=\mu_o$ ($\epsilon_r=1$, $\mu_r=1$). We may specialize the lossless medium equations for the case of free space.

$$\alpha = 0 \qquad\qquad \beta = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r} = \frac{\omega}{c}$$

$$u = \frac{\omega}{\beta} = \frac{c}{\sqrt{\mu_r\epsilon_r}} = c \qquad\qquad \lambda = \frac{2\pi}{\beta} = \frac{c}{f}$$

$$\eta = \eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 377\ \Omega$$

Wave Propagation in Good Conductors ($\sigma \gg \omega\epsilon$)

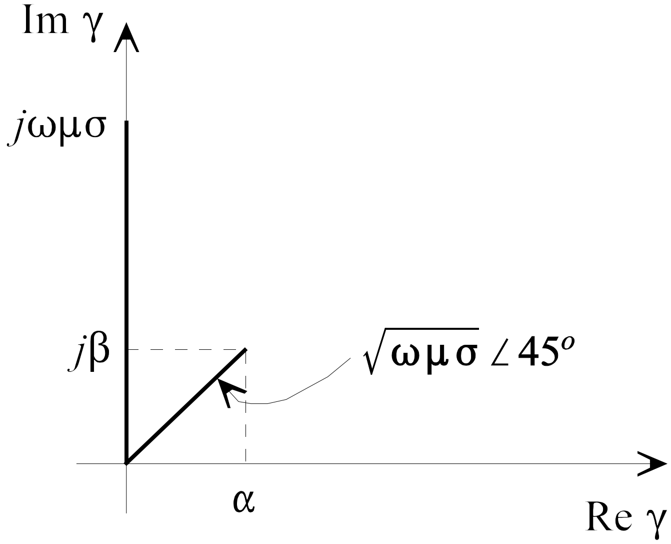
In a good conductor, displacement current is negligible in comparison to conduction current.

$$\boldsymbol{J}_{total} = \boldsymbol{J}_{conduction} + \boldsymbol{J}_{displacement} = \sigma\boldsymbol{E} + j\omega\epsilon\boldsymbol{E}$$

$$|\boldsymbol{J}_{conduction}| \gg |\boldsymbol{J}_{displacement}| \qquad \text{if} \qquad (\sigma \gg \omega\epsilon)$$

Although this inequality is frequency dependent, most good conductors (such as copper and aluminum) have conductivities on the order of 10^7 ℧/m and negligible polarization ($\epsilon_r=1$, $\epsilon=\epsilon_o=8.854\times10^{-12}$ F/m) such that we never encounter the frequencies at which the displacement current becomes comparable to the displacement current. Given $\sigma \gg \omega\epsilon$, the propagation constant within a good conductor may be approximated by

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} \approx \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma\angle 90^o} = \sqrt{\omega\mu\sigma}\angle 45^o$$

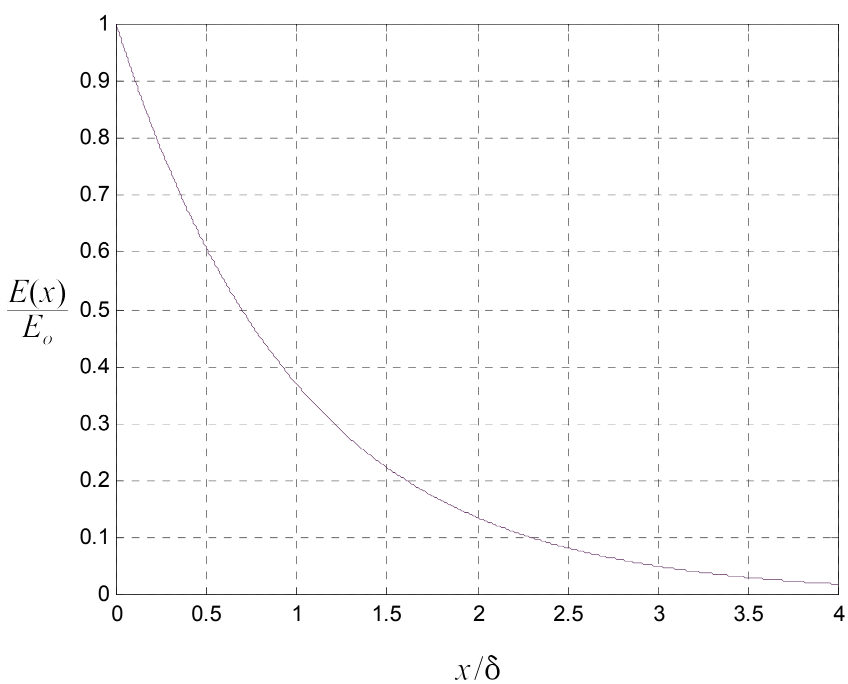


$$\gamma = \alpha + j\beta = \sqrt{\frac{\omega\mu\sigma}{2}}\,(1+j)$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$$

Note that attenuation in a good conductor increases with frequency. The rate of attenuation in a good conductor can be characterized by a distance defined as the *skin depth*.

> Skin depth (δ) − distance over which a plane wave is attenuated by a factor of e^{-1} in a good conductor.



$$E(x) = E_o e^{-\alpha x}$$

$$E(\delta) = E_o e^{-\alpha\delta}$$

$$= E_o e^{-1}$$

$$\alpha\delta = 1$$

$$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu\sigma}}$$

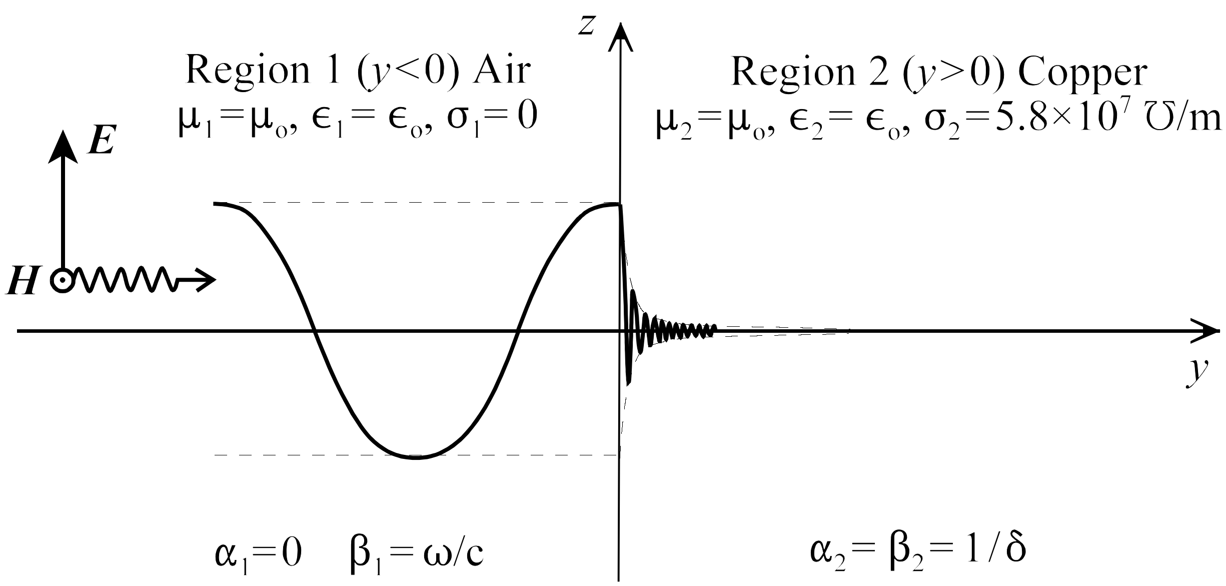
The velocity of propagation, wavelength and intrinsic impedance within the good conductor is

$$u = \frac{\omega}{\beta} = \frac{\omega\sqrt{2}}{\sqrt{\omega\mu\sigma}} = \sqrt{\frac{2\omega}{\mu\sigma}} \qquad \lambda = \frac{2\pi}{\beta}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \approx \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^o$$

<u>Example</u> (skin depth)

Uniform plane wave ($f = 1$ MHz) at an air/copper interface.



In the air,

$$\beta_1 = \frac{\omega}{c} = \frac{2\pi 10^6}{3\times 10^8} = 0.0209 \text{ rad/m} \qquad \lambda_1 = \frac{c}{f} = \frac{3\times 10^8}{10^6} = 300 \text{ m}$$

$$u_1 = c = 3\times 10^8 \text{ m/s}$$

In the copper,

$$\delta = \frac{1}{\sqrt{\pi f \mu_o \sigma}} = \frac{1}{\sqrt{\pi f (4\pi 10^{-7})(5.8\times 10^{7})}} = \frac{0.066}{\sqrt{f}}\ \text{m}$$

$$\text{at 1 MHz} \qquad \Rightarrow \qquad \delta = 0.066\ \text{mm}$$

$$\alpha_2 = \beta_2 = \frac{1}{\delta} = \frac{1}{0.066\times 10^{-3}\ \text{m}} = 15.2\times 10^{3}\ \text{Np/m, rad/m}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = 2\pi\delta = 2\pi(0.066\ \text{mm}) = 0.415\ \text{mm}$$

$$u_2 = \lambda_2 f = 415\ \text{m/s}$$

<u>Electromagnetic Shielding</u>

The previous results show that we may enclose a volume with a thin layer of good conductor to act as an electromagnetic shield. Depending on the application, the electromagnetic shield may be necessary to prevent waves from radiating out of the shielded volume or to prevent waves from penetrating into the shielded volume.

Skin Effect

Given a plane wave incident on a highly-conducting surface, the electric field (and thus current density) was found to be concentrated at the surface of the conductor. The same phenomenon occurs for a current carrying conductor such as a wire. The effect is frequency-dependent, just as it is in the incident plane wave example. This phenomenon is known as the *skin effect*.

Maxwell's curl equations for a time-harmonic current in a good conductor are

$$\nabla \times \boldsymbol{E}_s = -j\omega\mu \boldsymbol{H}_s \qquad ①$$

$$\nabla \times \boldsymbol{H}_s = \boldsymbol{J}_s + \cancelto{0}{j\omega\epsilon \boldsymbol{E}_s} \approx \boldsymbol{J}_s = \sigma \boldsymbol{E}_s \qquad ②$$

where the displacement current is assumed to be negligible in the good conductor. We would like to determine the governing PDE for the current density within the conductor. If we take the divergence of ②, we find

$$\nabla \cdot \nabla \times \boldsymbol{H}_s = \nabla \cdot \boldsymbol{J}_s = 0 \qquad (\nabla \cdot \nabla \times \boldsymbol{F} = 0 \quad \text{for any vector } \mathbf{F})$$

If we then take the curl of ①, we find

$$\nabla \times \nabla \times \boldsymbol{E}_s = -j\omega\mu \nabla \times \boldsymbol{H}_s$$

Using ②, we may write both the phasor electric and magnetic fields in terms of the current density.

$$\boldsymbol{E}_s = \frac{\boldsymbol{J}_s}{\sigma} \qquad \nabla \times \boldsymbol{H}_s = \boldsymbol{J}_s$$

$$\nabla \times \nabla \times \left(\frac{\boldsymbol{J}_s}{\sigma} \right) = -j\omega\mu \boldsymbol{J}_s$$

$$\nabla \times \nabla \times \boldsymbol{J}_s = -j\omega\mu\sigma \boldsymbol{J}_s$$

$$\nabla\times\nabla\times\boldsymbol{F} = \nabla(\nabla\cdot\boldsymbol{F}) - \nabla^2\boldsymbol{F} \qquad \text{(vector identity)}$$

$$\nabla\times\nabla\times\boldsymbol{J} = \nabla(\cancelto{0}{\nabla\cdot\boldsymbol{J}}) - \nabla^2\boldsymbol{J} = -\nabla^2\boldsymbol{J}$$

$$\nabla^2\boldsymbol{J}_s - j\omega\mu\sigma\boldsymbol{J}_s = 0$$

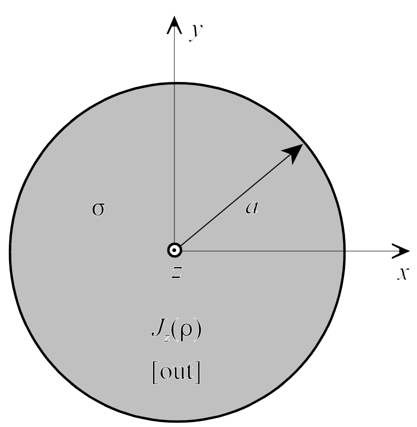
If we let $-j\omega\mu\sigma = T^2$, the governing equation for the conductor current density becomes

$$\nabla^2\boldsymbol{J}_s + T^2\boldsymbol{J}_s = 0 \qquad \left(\begin{array}{c}\text{vector wave}\\ \text{equation}\end{array}\right) \qquad ③$$

The constant T in the vector wave equation may be written in terms of the skin depth of the conductor.

$$T = \sqrt{-j\omega\mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{j}} = j^{-1/2}\sqrt{2\pi f\mu\sigma} = j^{-1/2}\frac{\sqrt{2}}{\delta}$$

For the special case of a cylindrical conductor (radius = a) lying along the z-axis, assuming only a z-component of current density which does not vary with respect to ϕ or z, the wave equation of ③ (in cylindrical coordinates) becomes



$$\frac{d^2J_{zs}(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dJ_{zs}(\rho)}{d\rho} + T^2J_{zs}(\rho) = 0$$

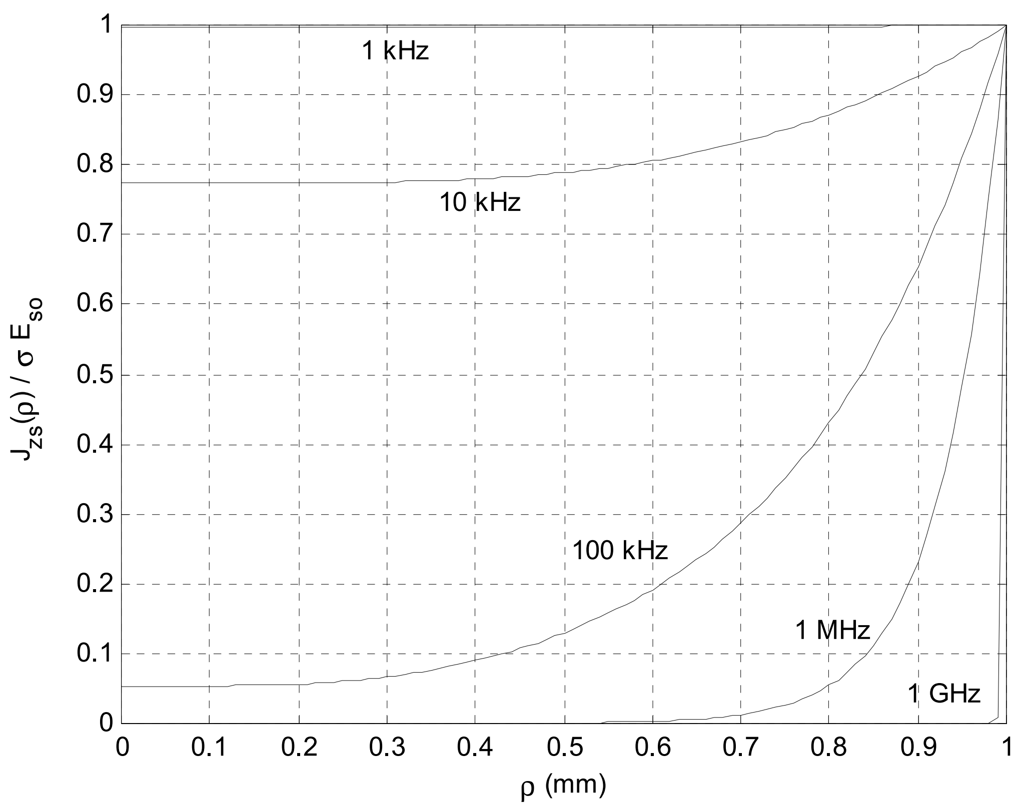
The differential equation governing the wire current density is *Bessel's differential equation* of order zero. The solution to the differential equation may be written in terms of *Bessel functions*.

solution $\Rightarrow$ $$J_{zs}(\rho) = \sigma E_{so} \frac{J_0(T\rho)}{J_0(Ta)}$$

σ - wire conductivity
a - wire radius
E_{so} - electric field at the wire surface
σE_{so} - current density at the wire surface
J_0 - Bessel function of the first kind, order 0

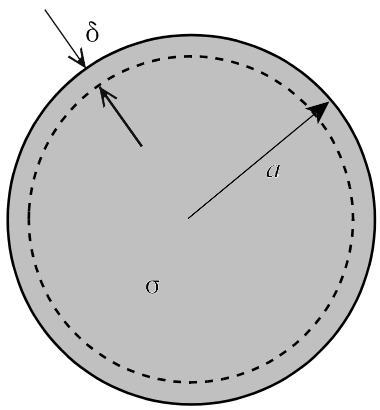
Example

Consider a copper wire of 1 mm radius. Plot the magnitude of the current density as a function of ρ at f = 1 kHz, 10 kHz, 100 kHz, 1 MHz, and 1 GHz.



As frequency increases, the current becomes concentrated along the outer surface of the wire.

The following is an equivalent model of a conducting wire at high frequency ($\delta \ll a$).



The equivalent model is a ring of uniform current density along the outer surface of the conductor (depth equal to one skin depth).

$$R_{DC} = \frac{l}{\sigma A_{DC}} = \frac{l}{\sigma \pi a^2} \qquad \text{(DC resistance)}$$

The DC resistance formula is only valid if the current density if uniform. We may use the DC resistance formula for the high frequency model of the conductor.

$$R_{AC} = \frac{l}{\sigma A_{AC}} = \frac{l}{\sigma[\pi a^2 - \pi(a-\delta)^2]} \qquad \begin{pmatrix} \text{high frequency} \\ \text{AC resistance} \end{pmatrix}$$

$$= \frac{l}{\sigma \pi [a^2 - a^2 + 2a\delta - \delta^2]}$$

$$\approx \frac{l}{\sigma(2\pi a)\delta}$$

Note that the high frequency AC resistance could have been found by folding out the skin depth cross-section around the perimeter of the wire into an approximate rectangular cross section given by a length of $2\pi a$ and a height of δ.

Complex Permittivity

The total current on the right hand side of Ampere's law consists of a conduction current term and a displacement current term. These two terms can be combined into one using the concept of a complex-valued permittivity.

$$\nabla \times \boldsymbol{H}_s = \sigma \boldsymbol{E}_s + j\omega\epsilon \boldsymbol{E}_s \qquad (\epsilon = \epsilon_r \epsilon_o)$$

$$= (\sigma + j\omega\epsilon)\boldsymbol{E}_s$$

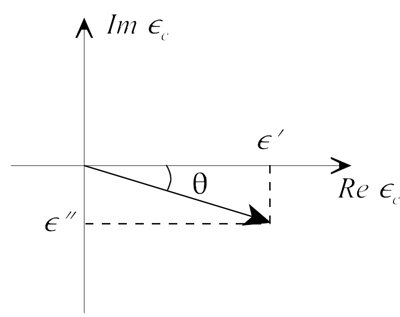
$$= j\omega\epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right)\boldsymbol{E}_s$$

$$= \underbrace{j\omega\epsilon\left[1 - j\left(\frac{\sigma}{\omega\epsilon}\right)\right]}_{\text{define as complex permittivity } \epsilon_c}\boldsymbol{E}_s$$

$$\nabla \times \boldsymbol{H}_s = j\omega\epsilon_c \boldsymbol{E}_s$$

$$\epsilon_c = \epsilon\left[1 - j\left(\frac{\sigma}{\omega\epsilon}\right)\right] = \epsilon' - j\epsilon'' = |\epsilon_c|\, e^{-j\theta}$$

$$\epsilon' = \epsilon \qquad \epsilon'' = \frac{\sigma}{\omega}$$



The ratio of the imaginary part of the complex permittivity (ϵ'') to the real part of the complex permittivity (ϵ') is the ratio of the magnitude of the conduction current density to the magnitude of the displacement current density. This ratio is defined as the *loss tangent* of the medium.

$$\frac{|\boldsymbol{J}_{conduction}|}{|\boldsymbol{J}_{displacement}|} = \frac{|\sigma \boldsymbol{E}_s|}{|j\omega\epsilon \boldsymbol{E}_s|} = \frac{\sigma}{\omega\epsilon} = \frac{\epsilon''}{\epsilon'} = \tan\theta \qquad \text{(loss tangent)}$$

Poynting's Theorem and the Poynting Vector

Poynting's theorem is the fundamental energy-conservation theorem for electromagnetic fields. Using Poynting's theorem, we can identify all sources of energy related to electromagnetic fields in a given volume. The corresponding *Poynting vector* defines the vector power density (direction and density of power flow at a point). To derive Poynting's theorem, we start with the time-dependent Maxwell curl equations.

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad ①$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad ②$$

The product of $\boldsymbol{E}$ and $\boldsymbol{H}$ gives units of W/m^2 (volume power density, analogous to volume current density). As shown for the uniform plane wave, the direction of $\boldsymbol{E}\times\boldsymbol{H}$ gives the direction of wave propagation (the direction of power flow). Thus, we seek a relationship defining the cross product of $\boldsymbol{E}$ and $\boldsymbol{H}$. Using the vector identity,

$$\nabla \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\nabla \times \boldsymbol{B}) \qquad ③$$

letting $\boldsymbol{A}=\boldsymbol{E}$ and $\boldsymbol{B}=\boldsymbol{H}$, we may obtain the necessary terms on the right hand side of ③ by dotting ① with $\boldsymbol{H}$ and ② with $\boldsymbol{E}$.

$$\boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \qquad ④$$

$$\boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) = \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \qquad ⑤$$

Inserting ④ and ⑤ into ③ yields

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} - \boldsymbol{E} \cdot \boldsymbol{J} \qquad ⑥$$

The three terms on the right hand side of ⑥ may be rewritten as

$$\boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t} = \mu\left(\boldsymbol{H}\cdot\frac{\partial \boldsymbol{H}}{\partial t}\right) = \mu\,\frac{1}{2}\,\frac{\partial}{\partial t}\left(\boldsymbol{H}\cdot\boldsymbol{H}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t} = \epsilon\left(\boldsymbol{E}\cdot\frac{\partial \boldsymbol{E}}{\partial t}\right) = \epsilon\,\frac{1}{2}\,\frac{\partial}{\partial t}\left(\boldsymbol{E}\cdot\boldsymbol{E}\right) = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2\right)$$

$$\boldsymbol{E}\cdot\boldsymbol{J} = \sigma\left(\boldsymbol{E}\cdot\boldsymbol{E}\right) = \sigma E^2$$

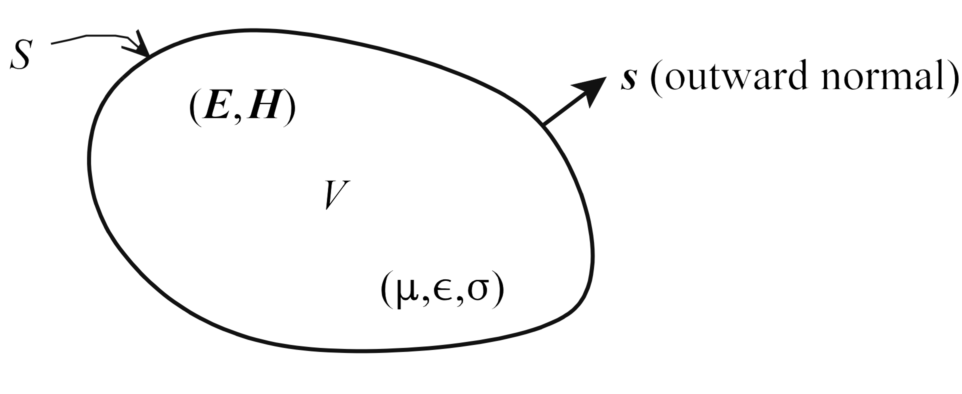
which gives

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2 \qquad ⑦$$

Integrating ⑦ over a given volume V (enclosed by a surface S) and applying the divergence theorem yields

$$\int_V \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})\,dv = \int_V \left\{-\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2\right\}dv$$

$$\underbrace{\int_S (\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{s}}_{\text{net power flow out of } V} = \underbrace{-\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)dv}_{\text{decrease in the stored electric and magnetic energy within } V} \underbrace{- \int_V \sigma E^2\,dv}_{\text{ohmic losses within } V} \qquad \begin{pmatrix}\text{Poynting's}\\ \text{theorem}\end{pmatrix}$$



The quantity under the integrand on the left hand side of Poynting's theorem is defined as the Poynting vector.

$$\wp = \boldsymbol{E} \times \boldsymbol{H} \qquad \begin{pmatrix} \text{instantaneous Poytning vector - direction} \\ \text{and density of power flow at a point} \end{pmatrix}$$

For a time-harmonic field, the time average Poynting vector is found by integrating the instantaneous Poynting vector over one period and dividing by the period.

$$\wp_{ave} = \frac{1}{T}\int_T (\boldsymbol{E} \times \boldsymbol{H})\, dt \qquad \begin{pmatrix} \text{time-average of the} \\ \text{instantaneous Poytning vector} \end{pmatrix}$$

The time-average Poynting vector can actually be determined without integrating if we use phasors. If we write the instantaneous electric and magnetic fields as

$$\boldsymbol{E} = |\boldsymbol{E}|\cos(\omega t + \theta_E)\,\boldsymbol{a}_E \qquad \boldsymbol{H} = |\boldsymbol{H}|\cos(\omega t + \theta_H)\,\boldsymbol{a}_H$$

then the instantaneous Poynting vector is

$$\wp = \boldsymbol{E} \times \boldsymbol{H} = |\boldsymbol{E}|\,|\boldsymbol{H}|\cos(\omega t + \theta_E)\cos(\omega t + \theta_H)\,(\boldsymbol{a}_E \times \boldsymbol{a}_H)$$

Using the trigonometric identity,

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

the instantaneous Poynting vector can be written as

$$\boldsymbol{E} \times \boldsymbol{H} = \frac{1}{2}\,|\boldsymbol{E}|\,|\boldsymbol{H}|\left[\cos(2\omega t + \theta_E + \theta_H) + \cos(\theta_E - \theta_H)\right](\boldsymbol{a}_E \times \boldsymbol{a}_H)$$

The time-average Poynting vector is then

$$\wp_{ave} = \frac{|\boldsymbol{E}|\,|\boldsymbol{H}|}{2T}(\boldsymbol{a}_E \times \boldsymbol{a}_H)\left\{\int_T \cos(2\omega t + \theta_E + \theta_H^{0})\, dt + \cos(\theta_E - \theta_H)\int_T dt\right\}$$

The time-average Poynting vector reduces to

$$\wp_{ave} = \frac{|\boldsymbol{E}|\,|\boldsymbol{H}|}{2T}\,(\boldsymbol{a}_E \times \boldsymbol{a}_H)\cos(\theta_E - \theta_H)\,T$$

$$= \frac{1}{2}\,Re\left[\left(|\boldsymbol{E}|\,e^{j\theta_E}\,\boldsymbol{a}_E\right) \times \left(|\boldsymbol{H}|\,e^{-j\theta_H}\,\boldsymbol{a}_H\right)\right]$$

$$= \frac{1}{2}\,Re\left[\boldsymbol{E}_s \times \boldsymbol{H}_s^*\right] \qquad ⑧$$

Note that the time-average Poynting vector above is determined without integration. The equation above is the vector analog of the time-average power equation used in circuit (phasor) analysis:

$$P_{ave} = \frac{1}{2}\,Re\left[V_s\,I_s^*\right]$$

The term in brackets in ⑧ is defined as the phasor Poynting vector and is normally represented by ***S***.

$$\boldsymbol{S} = \boldsymbol{E}_s \times \boldsymbol{H}_s^* \qquad \text{(phasor Poynting vector)}$$

$$\wp_{ave} = \frac{1}{2}\,Re[\boldsymbol{S}]$$

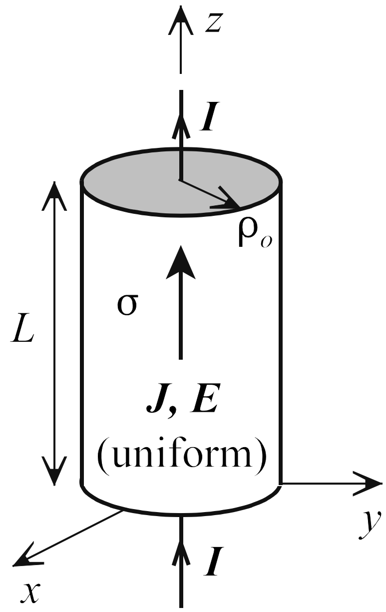
All representations of the Poynting vector represent vector energy densities. Thus, to determine the total power passing through a surface, we must integrate the Poynting vector over that surface. The total time-average power passing through the surface S is

$$P_{ave} = \int_S \wp_{ave} \cdot d\boldsymbol{s} \qquad \left(\begin{array}{c}\text{total time-average power}\\ \text{passing through the surface S}\end{array}\right)$$

$$= \frac{1}{2}\,Re\int_S \left[\boldsymbol{E}_s \times \boldsymbol{H}_s^*\right] \cdot d\boldsymbol{s} = \frac{1}{2}\,Re\int_S \boldsymbol{S} \cdot d\boldsymbol{s}$$

<u>Example</u> (Poynting vector / power flow in a resistor)

Assume a DC current I in an ideal resistor as shown below. Determine the instantaneous Poynting vector, the time-average power, and the total power dissipated in the resistor.



$$\boldsymbol{J} = \sigma \boldsymbol{E} = \frac{\boldsymbol{I}}{A} = \frac{I}{\pi \rho_o^2}\boldsymbol{a}_z \qquad\qquad \boldsymbol{E} = \frac{I}{\sigma \pi \rho_o^2}\boldsymbol{a}_z$$

From Ampere's law, the magnetic field in the resistor is

$$\boldsymbol{H} = H_\phi \boldsymbol{a}_\phi = \frac{I\rho}{2\pi\rho_o^2}\boldsymbol{a}_\phi \qquad (\rho < \rho_o)$$

The instantaneous Poynting vector is

$$\wp = \boldsymbol{E} \times \boldsymbol{H} = \frac{I}{\sigma \pi \rho_o^2}\boldsymbol{a}_z \times \frac{I\rho}{2\pi\rho_o^2}\boldsymbol{a}_\phi = \frac{I^2 \rho}{2\sigma\pi^2\rho_o^4}(-\boldsymbol{a}_\rho)$$

The time-average Poynting vector is

$$\wp_{ave} = \frac{1}{T}\int_T \wp \, dt = \wp \qquad\qquad (\text{time} - \text{average} = \text{instantaneous for DC})$$

The total power dissipated in the resistor is found by integrating the power flow **into** the resistor volume.

$$P_{ave} = \int_S \wp_{ave} \cdot d\boldsymbol{s} = \int_{\text{cylinder}} \wp_{ave} \cdot d\boldsymbol{s} + \int_{\text{endcaps}} \wp_{ave} \cdot d\boldsymbol{s}$$

$$d\boldsymbol{s} = \rho_o \, d\phi \, dz \, (-\boldsymbol{a}_\rho) \qquad d\boldsymbol{s} = \rho \, d\rho \, d\phi \, (\pm \boldsymbol{a}_z)$$

$$P_{ave} = \int_0^L \int_0^{2\pi} \underbrace{\left[\frac{I^2}{2\sigma\pi^2\rho_o^3}(-\boldsymbol{a}_\rho) \right]}_{\wp_{\text{ave}}(\rho=\rho_o)} \cdot \left[\rho_o \, d\phi \, dz(-\boldsymbol{a}_\rho) \right]$$

$$= \frac{I^2}{2\sigma\pi^2\rho_o^2} \int_0^L \int_0^{2\pi} d\phi \, dz = \frac{I^2}{2\sigma\pi^2\rho_o^2}(L)(2\pi)$$

$$= I^2 \left(\frac{L}{\sigma\pi\rho_o^2} \right) = I^2 \frac{L}{\sigma A} = I^2 R \quad (\text{W})$$

The total power dissipated in the resistor is equal to the power flow into the volume of the resistor.

Plane Wave Reflection/Transmission at a Dielectric Interface

When a plane wave propagating in a homogenous medium encounters an interface with a different medium, a portion of the wave is reflected from the interface while the remainder of the wave is transmitted. The reflected and transmitted waves can be determined by enforcing the fundamental electromagnetic field boundary conditions at the media interface.

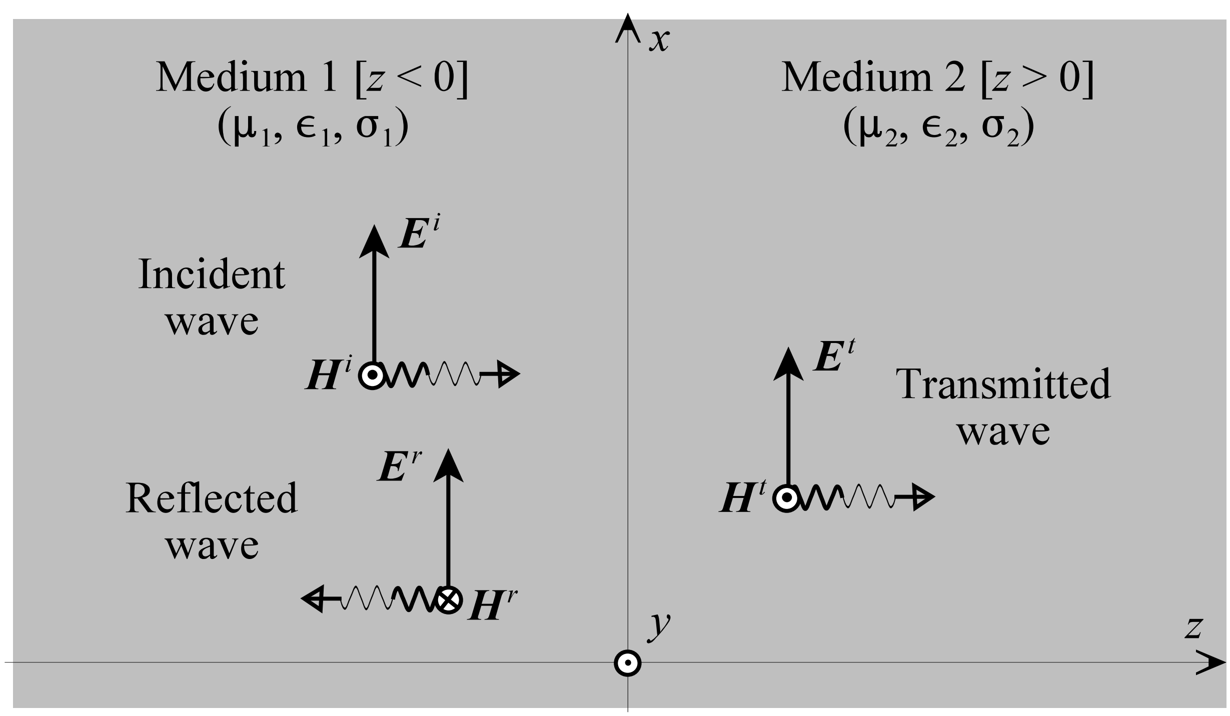
Given a z-directed, x-polarized uniform plane wave incident on a planar media interface located on the x-y plane, the phasor fields associated with the incident, reflected and transmitted fields may be written as

Incident wave fields
$$\begin{cases} \boldsymbol{E}_s^i = E_{so} e^{-\gamma_1 z} \boldsymbol{a}_x \\ \boldsymbol{H}_s^i = \dfrac{E_{so}}{\eta_1} e^{-\gamma_1 z} \boldsymbol{a}_y \end{cases}$$

Reflected wave fields
$$\begin{cases} \boldsymbol{E}_s^r = \Gamma E_{so} e^{\gamma_1 z} \boldsymbol{a}_x \\ \boldsymbol{H}_s^r = -\Gamma \dfrac{E_{so}}{\eta_1} e^{\gamma_1 z} \boldsymbol{a}_y \end{cases}$$

Transmitted wave fields
$$\begin{cases} \boldsymbol{E}_s^t = \tau E_{so} e^{-\gamma_2 z} \boldsymbol{a}_x \\ \boldsymbol{H}_s^t = \tau \dfrac{E_{so}}{\eta_2} e^{-\gamma_2 z} \boldsymbol{a}_y \end{cases}$$

Γ - Reflection coefficient
τ - Transmission coefficient



Enforcement of the boundary conditions (continuous tangential electric field and continuous tangential magnetic field) yields

$$E_x^i + E_x^r = E_x^t \qquad \text{at} \quad z = 0 \qquad \Rightarrow \qquad 1 + \Gamma = \tau$$

$$H_y^i + H_y^r = H_y^t \qquad \text{at} \quad z = 0 \qquad \Rightarrow \qquad \frac{1 - \Gamma}{\eta_1} = \frac{\tau}{\eta_2}$$

Solving these two equations for the reflection and transmission coefficients gives

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \qquad \text{(reflection coefficient)}$$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} \qquad \text{(transmission coefficient)}$$

The total fields in the two media are

$$\left.\begin{aligned} \boldsymbol{E}_1 &= \boldsymbol{E}_s^i + \boldsymbol{E}_s^r = E_{so}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\boldsymbol{a}_x \\ \boldsymbol{H}_1 &= \boldsymbol{H}_s^i + \boldsymbol{H}_s^r = \frac{E_{so}}{\eta_1}\left(e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right)\boldsymbol{a}_y \end{aligned}\right\} \qquad \left(\begin{array}{c}\text{total fields}\\ \text{in region 1}\end{array}\right)$$

$$\left.\begin{aligned} \boldsymbol{E}_2 &= \boldsymbol{E}_s^t = E_{so}\tau e^{-\gamma_2 z}\,\boldsymbol{a}_x \\ \boldsymbol{H}_2 &= \boldsymbol{H}_s^t = \frac{E_{so}}{\eta_2}\tau e^{-\gamma_2 z}\,\boldsymbol{a}_y \end{aligned}\right\} \qquad \left(\begin{array}{c}\text{total fields}\\ \text{in region 2}\end{array}\right)$$

<u>Special cases</u>

$$\eta_1 = \eta_2 \qquad \Gamma = 0 \qquad \tau = 1 \qquad \left(\begin{array}{c}\text{total transmission}\\ \text{no reflection}\end{array}\right)$$

$$\eta_1 = 0 \qquad \Gamma = 1 \qquad \tau = 2 \qquad \left(\begin{array}{c}\text{total reflection}\\ \text{without inversion of } \boldsymbol{E}\end{array}\right)$$

$$\eta_2 = 0 \qquad \Gamma = -1 \qquad \tau = 0 \qquad \left(\begin{array}{c}\text{total reflection}\\ \text{with inversion of } \boldsymbol{E}\end{array}\right)$$

Special Case #1

$$\text{region 1} \;\Rightarrow\; \text{lossless dielectric} \left[\sigma_1 = 0,\ \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}},\ \alpha_1 = 0,\ \gamma = j\beta_1\right]$$

$$\text{region 2} \;\Rightarrow\; \text{perfect conductor} \left[\sigma_2 = \infty,\ \eta_2 = 0,\ \alpha_2 = \beta_2 \rightarrow \infty\right]$$

$$\eta_2 = 0 \qquad \Gamma = -1 \qquad \tau = 0 \qquad \left(\begin{array}{c}\text{total reflection}\\ \text{with inversion of } \boldsymbol{E}\end{array}\right)$$

$$\boldsymbol{E}_{1s} = E_{so}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\boldsymbol{a}_x = E_{so}\left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right)\boldsymbol{a}_x = -2jE_{so}\sin\beta_1 z\,\boldsymbol{a}_x$$

$$\boldsymbol{H}_{1s} = \frac{E_{so}}{\eta_1}\left(e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right)\boldsymbol{a}_y = \frac{E_{so}}{\eta_1}\left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right)\boldsymbol{a}_y = \frac{2E_{so}}{\eta_1}\cos\beta_1 z\,\boldsymbol{a}_y$$

$$\boldsymbol{E}_{2s} = E_{so}\,\tau\, e^{-\gamma_2 z}\,\boldsymbol{a}_x = \boldsymbol{0}$$

$$\boldsymbol{H}_{2s} = \frac{E_{so}}{\eta_2}\,\tau\, e^{-\gamma_2 z}\,\boldsymbol{a}_y = \boldsymbol{0}$$

If we let $E_{so} = |E_{so}|e^{j\phi}$ and $-j = e^{-j(\pi/2)}$, the instantaneous fields in the dielectric are

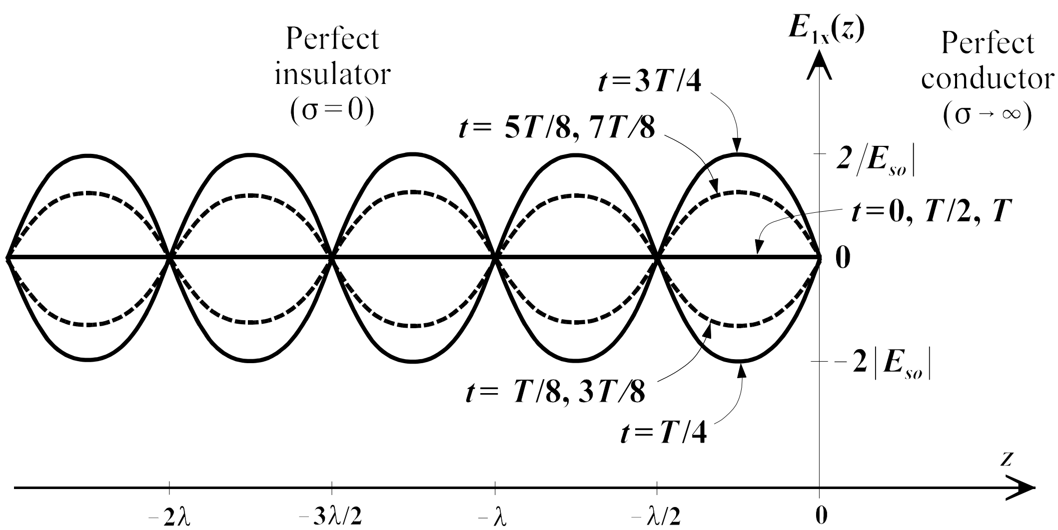
$$\begin{aligned}
\boldsymbol{E}_1 &= Re\left[\boldsymbol{E}_{1s}\,e^{j\omega t}\right] = Re\left[2e^{-j\pi/2}|E_{so}|e^{j\phi}\sin\beta_1 z\,e^{j\omega t}\boldsymbol{a}_x\right]\\
&= 2\,|E_{so}|\,\sin\beta_1 z\cos(\omega t + \phi - \pi/2)\,\boldsymbol{a}_x\\
&= 2\,|E_{so}|\,\sin\beta_1 z\sin(\omega t + \phi)\,\boldsymbol{a}_x
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{H}_1 &= Re\left[\boldsymbol{H}_{1s}\,e^{j\omega t}\right] = Re\left[\frac{2\,|E_{so}|}{\eta_1}\,e^{j\phi}\cos\beta_1 z\,e^{j\omega t}\boldsymbol{a}_y\right]\\
&= \frac{2\,|E_{so}|}{\eta_1}\cos\beta_1 z\cos(\omega t + \phi)\,\boldsymbol{a}_y
\end{aligned}$$

Note that the position dependence of the instantaneous electric and magnetic fields is not a function of time (standing wave).

Assuming for simplicity that $\phi = 0^{\circ}$ (the phase of the incident electric field is 0° at the media interface), the instantaneous electric field in the dielectric is

$$\boldsymbol{E}_1 = 2\,|E_{so}|\,\sin\beta_1 z\,\sin\omega t\,\boldsymbol{a_x}$$



The locations of the minimums and maximums of the standing wave electric field pattern are found by

$$|\boldsymbol{E}_1|_{\min} \text{ occurs when } \beta_1(-z) = n\pi \;\Rightarrow\; z = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{(2\pi)/\lambda_1} = -n\frac{\lambda_1}{2}$$

$$n = 0, 1, 2, \ldots$$

$$|\boldsymbol{E}_1|_{\max} \text{ occurs when } \beta_1(-z) = (2n+1)\frac{\pi}{2} \;\Rightarrow\; z = -\frac{(2n+1)\pi}{2(2\pi/\lambda_1)} = -(2n+1)\frac{\lambda_1}{4}$$

$$n = 0, 1, 2, \ldots$$

Special Case #2

$$\text{region 1} \;\Rightarrow\; \text{lossless dielectric} \quad \left[\sigma_1 = 0,\ \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}},\ \alpha_1 = 0,\ \gamma = j\beta_1\right]$$

$$\text{region 2} \;\Rightarrow\; \text{lossless dielectric} \quad \left[\sigma_2 = 0,\ \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}},\ \alpha_2 = 0,\ \gamma = j\beta_2\right]$$

$$\text{if } \eta_2 > \eta_1 \qquad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma \angle 0^o \qquad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \tau \angle 0^o$$

$$(0 < \Gamma < 1) \qquad (1 < \tau < 2)$$

$$\text{if } \eta_2 < \eta_1 \qquad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma \angle 180^o \qquad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \tau \angle 0^o$$

$$(0 < \Gamma < 1) \qquad (0 < \tau < 1)$$

$$\boldsymbol{E}_{1s} = E_{so}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\boldsymbol{a_x} = E_{so}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)\boldsymbol{a_x}$$

$$= E_{so}\, e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right)\boldsymbol{a_x}$$

$$\boldsymbol{H}_{1s} = \frac{E_{so}}{\eta_1}\left(e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}\right)\boldsymbol{a_y} = \frac{E_{so}}{\eta_1}\left(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}\right)\boldsymbol{a_y}$$

$$= \frac{E_{so}}{\eta_1} e^{-j\beta_1 z}\left(1 - \Gamma e^{j2\beta_1 z}\right)\boldsymbol{a_y}$$

$$\boldsymbol{E}_{2s} = E_{so}\,\tau\, e^{-\gamma_2 z}\,\boldsymbol{a_x} = E_{so}\,\tau\, e^{-j\beta_2 z}\,\boldsymbol{a_x}$$

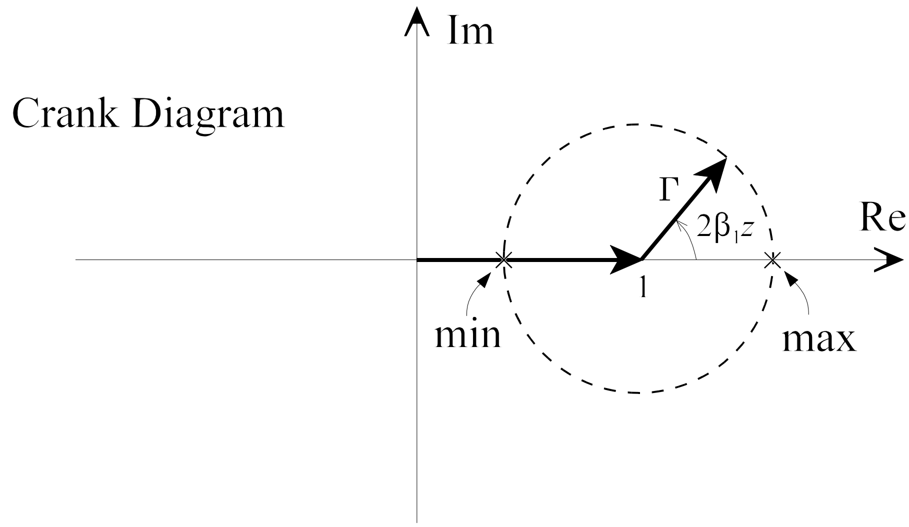
$$\boldsymbol{H}_{2s} = \frac{E_{so}}{\eta_2}\,\tau\, e^{-\gamma_2 z}\,\boldsymbol{a_y} = \frac{E_{so}}{\eta_2}\,\tau\, e^{-j\beta_2 z}\,\boldsymbol{a_y}$$

Note that standing waves exist only in region 1.

The magnitude of the electric field in region 1 can be analyzed to determine the locations of the maximum and minimum values of the standing wave electric field.

$$|\boldsymbol{E}_{1s}| = |E_{so}| \left| 1 + \Gamma e^{j2\beta_1 z} \right|$$

$$\left| 1 + \Gamma e^{j2\beta_1 z} \right| = \left| 1\angle 0^o + \Gamma \angle 2\beta_1 z \right|$$



If $\eta_2 > \eta_1$ (Γ is positive), then

$$\left| 1 + \Gamma e^{j2\beta_1 z} \right|_{\max} = 1 + |\Gamma| \quad \text{when} \quad 2\beta_1(-z) = n(2\pi) \quad n = 0, 1, 2, ...$$

$$z = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{(2\pi)/\lambda_1} = -n\frac{\lambda_1}{2}$$

$$\left| 1 + \Gamma e^{j2\beta_1 z} \right|_{\min} = 1 - |\Gamma| \quad \text{when} \quad 2\beta_1(-z) = (2n+1)\pi \qquad n = 0, 1, 2, ...$$

$$z = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)\pi}{2(2\pi)/\lambda_1} = -(2n+1)\frac{\lambda_1}{4}$$

$$|\boldsymbol{E}_{1s}|_{\max} = |E_{so}| \, (1 + |\Gamma|)$$

$$|\boldsymbol{E}_{1s}|_{\min} = |E_{so}| \, (1 - |\Gamma|)$$

If $\eta_1 > \eta_2$ (Γ is negative), and the positions of the maximums and minimums are reversed, but the equations for the maximum and minimum electric field magnitudes in terms of $|\Gamma|$ are the same.

The *standing wave ratio* (s) in a region where standing waves exist is defined as the ratio of the maximum electric field magnitude to the minimum electric field magnitude.

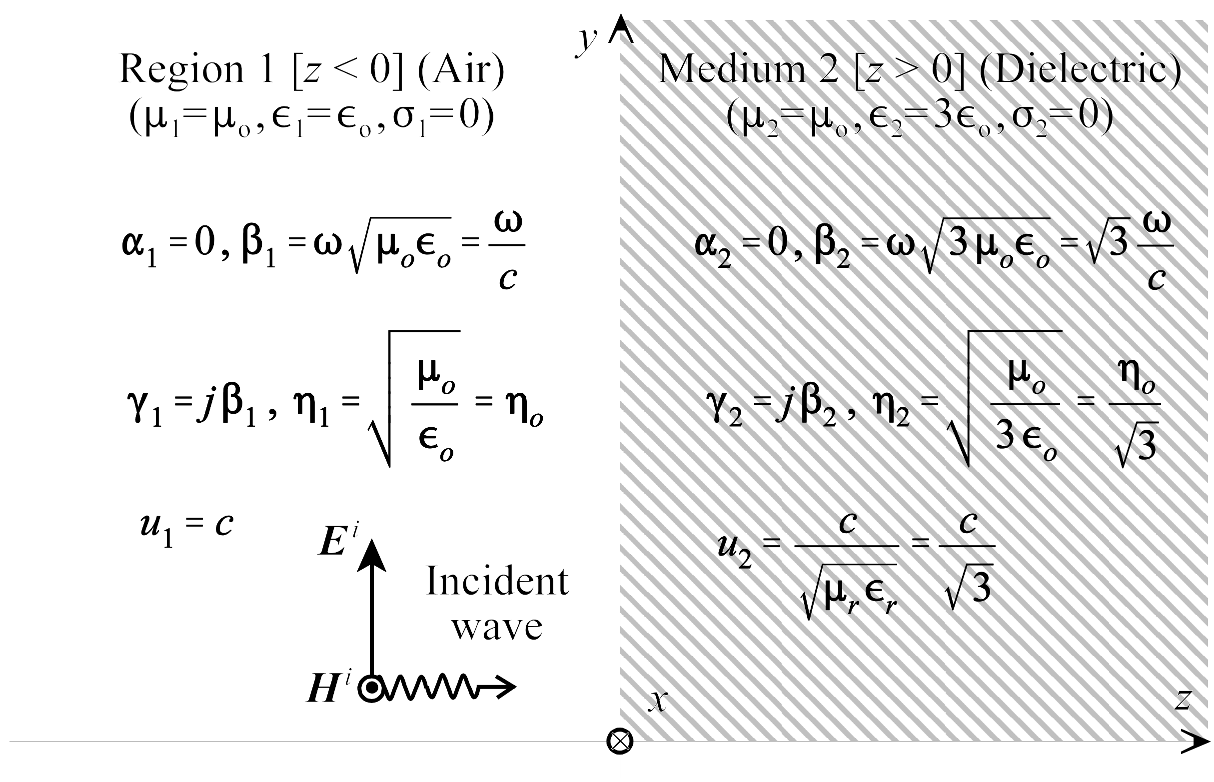
$$s = \frac{|\boldsymbol{E}_{1s}|_{max}}{|\boldsymbol{E}_{1s}|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad \Rightarrow \qquad |\Gamma| = \frac{s-1}{s+1}$$

The standing wave ratio (purely real) ranges from a minimum value of 1 (no reflection, $|\Gamma|=0$) to ∞ (total reflection, $|\Gamma|=1$). The standing wave ratio is sometimes defined is dB as

$$s(\mathrm{dB}) = 20\log_{10} s$$

<u>Example</u> (Plane wave reflection/transmission)

A uniform plane wave in air is normally incident on an infinite lossless dielectric material having $\epsilon=3\epsilon_o$ and $\mu=\mu_o$. If the incident wave is $\boldsymbol{E}_i=10\cos(\omega t - z)\,\boldsymbol{a}_y$ V/m, find (a.) ω and λ of the waves in both regions, (b.) $\boldsymbol{H}_i$, (c.) Γ and τ, (d.) The total electric field and time-average power in both regions.



(a.) $\boldsymbol{E}_i = 10\cos(\omega t - z)\,\boldsymbol{a}_y \quad \Rightarrow \quad \boldsymbol{E}_{is} = 10\,e^{-jz}\boldsymbol{a}_y = E_{so}\,e^{-j\beta_1 z}\boldsymbol{a}_y$

$$E_{so} = 10 \qquad \beta_1 = 1$$

$$\beta_1 = \frac{2\pi}{\lambda_1} = \frac{\omega}{u_1} = \frac{\omega}{c} = 1 \text{ rad/m} \qquad \beta_2 = \frac{2\pi}{\lambda_2} = \frac{\omega}{u_2} = \sqrt{3}\,\frac{\omega}{c} = \sqrt{3}\,\beta_1 \text{ rad/m}$$

$$\lambda_1 = \frac{2\pi}{\beta_1} = 2\pi = 6.28 \text{ m} \qquad \lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = 3.63 \text{ m}$$

$$\omega = \beta_1 u_1 = \beta_2 u_2 = 3\times 10^8 \text{ rad/s } (47.8\,\text{MHz})$$

(b.) $$\boldsymbol{H}_{is} = \frac{E_{so}}{\eta_o}\,e^{-j\beta_1 z}(-\boldsymbol{a}_x) = -\frac{10}{377}\,e^{-jz}\,\boldsymbol{a}_x = -0.0266\,e^{-jz}\,\boldsymbol{a}_x$$

$$\boldsymbol{H}_i = Re\left\{-0.0266\,e^{-jz}e^{j\omega t}\,\boldsymbol{a}_x\right\} = -0.0266\cos(\omega t - z)\,\boldsymbol{a}_x \text{ A/m}$$

(c.) $$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\dfrac{\eta_o}{\sqrt{3}} - \eta_o}{\dfrac{\eta_o}{\sqrt{3}} + \eta_o} = -0.268$$

$$\tau = 1 + \Gamma = 0.732$$

(d.) $$\boldsymbol{E}_{1s} = E_{so}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)\boldsymbol{a}_y = \left(10\,e^{-jz} - 2.68\,e^{jz}\right)\boldsymbol{a}_y$$

$$\boldsymbol{E}_{2s} = \tau\,E_{so}\,e^{-j\beta_2 z}\,\boldsymbol{a}_y = 7.32\,e^{-j\sqrt{3}z}\,\boldsymbol{a}_y$$

$$\boldsymbol{E}_1 = Re\left\{\left(10\,e^{-jz} - 2.68\,e^{jz}\right)e^{j\omega t}\boldsymbol{a}_y\right\} = [10\cos(\omega t - z) - 2.68\cos(\omega t + z)]\,\boldsymbol{a}_y \text{ V/m}$$

$$\boldsymbol{E}_2 = Re\left\{7.32\,e^{-j\sqrt{3}z}e^{j\omega t}\boldsymbol{a}_y\right\} = [7.32\cos(\omega t - \sqrt{3}z)]\,\boldsymbol{a}_y \text{ V/m}$$

In general, the power flow for a plane wave may be written as

$$\mathcal{P}_{ave} = \frac{1}{2} Re\left[\boldsymbol{E}_s \times \boldsymbol{H}_s^*\right] = \frac{1}{2} Re\left[E_s H_s^* \boldsymbol{a}_k\right]$$

$$H_s = \frac{E_s}{\eta} \qquad\qquad H_s^* = \frac{E_s^*}{\eta^*}$$

$$\mathcal{P}_{ave} = \frac{\boldsymbol{a}_k}{2} Re\left[\frac{E_s E_s^*}{\eta^*}\right] = \frac{\boldsymbol{a}_k}{2} Re\left[\frac{|E_s|^2}{|\eta| e^{-j\theta_\eta}}\right] = \frac{\boldsymbol{a}_k}{2} Re\left[\frac{|E_s|^2}{|\eta|} e^{j\theta_\eta}\right]$$

$$= \frac{1}{2} \frac{|E_s|^2}{|\eta|} \cos\theta_\eta \, \boldsymbol{a}_k$$

For a plane wave propagating in a lossless dielectric (η is real, $\theta_\eta = 0$), the power flow reduces to

$$\mathcal{P}_{ave} = \frac{|E_s|^2}{2\eta} \boldsymbol{a}_k$$

For this problem,

$$\mathcal{P}_{1,ave} = \frac{|E_{is}|^2}{2\eta_1} \boldsymbol{a}_z + \frac{|E_{rs}|^2}{2\eta_1} (-\boldsymbol{a}_z)$$

$$= \frac{(10)^2 - (2.68)^2}{2(377)} \boldsymbol{a}_z$$

$$= 123 \boldsymbol{a}_z \ \frac{\text{mW}}{\text{m}^2}$$

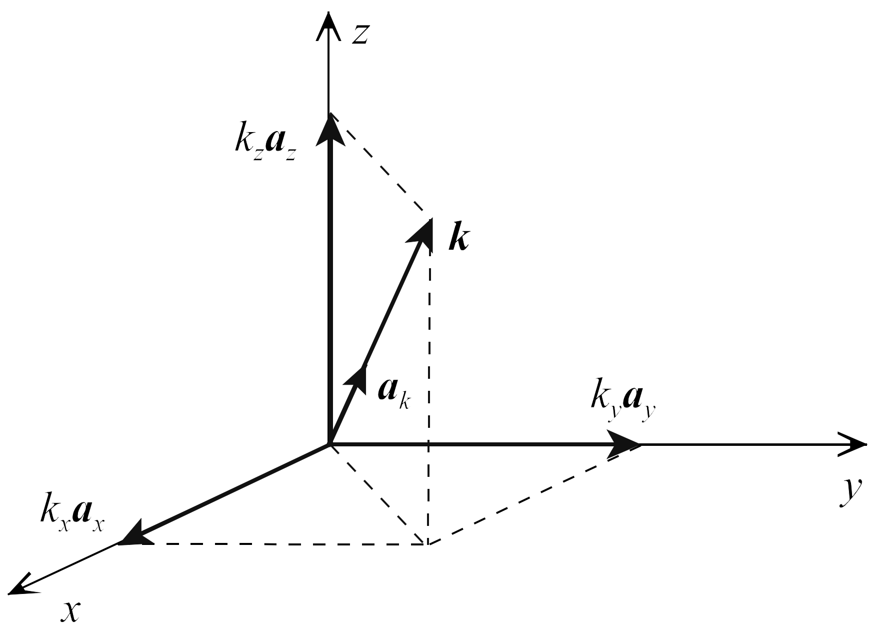
$$\mathcal{P}_{2,ave} = \frac{|E_{ts}|^2}{2\eta_1} \boldsymbol{a}_z = \frac{(7.32)^2}{2(377/\sqrt{3})} \boldsymbol{a}_z$$

$$= 123 \boldsymbol{a}_z \ \frac{\text{mW}}{\text{m}^2}$$

Arbitrarily Directed Plane Wave

In order to define an arbitrarily directed plane wave, we define a *vector wavenumber* or *propagation vector* (**k**).

$$\boldsymbol{k} = k\boldsymbol{a}_k = \omega\sqrt{\mu\epsilon}\,\boldsymbol{a}_k = k_x\boldsymbol{a}_x + k_y\boldsymbol{a}_y + k_z\boldsymbol{a}_z$$



The electric field of an $\boldsymbol{a}_z$-directed plane wave (in a lossless medium) may be written as

$$\boldsymbol{E}_s = E_{so}e^{-j\beta z}\,\boldsymbol{a}_E$$

The electric field of an $\boldsymbol{a}_k$-directed plane wave (in an arbitrary medium) may be written as

$$\boldsymbol{E}_s = E_{so}e^{-j(k_x x + k_y y + k_z z)}\,\boldsymbol{a}_E$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\epsilon \qquad \left\{\begin{array}{ll} \epsilon = \epsilon_o\epsilon_r & \text{(lossless)} \\ \epsilon = \epsilon_o(\epsilon' - j\epsilon'') & \text{(lossy)} \end{array}\right\}$$

The fields of an arbitrarily directed plane wave can be written concisely in terms of the dot product (scalar product) of the position vector $\boldsymbol{r}$ and the propagation vector $\boldsymbol{k}$.

$$\boldsymbol{k} = k_x \boldsymbol{a}_x + k_y \boldsymbol{a}_y + k_z \boldsymbol{a}_z$$

$$\boldsymbol{r} = x \boldsymbol{a}_x + y \boldsymbol{a}_y + z \boldsymbol{a}_z$$

$$\boldsymbol{E}_s = E_{so} e^{-j(\boldsymbol{k}\cdot\boldsymbol{r})} \boldsymbol{a}_E = E_{so} e^{-j(k_x x + k_y y + k_z z)} \boldsymbol{a}_E$$

$$\boldsymbol{H}_s = \frac{E_{so}}{\eta} e^{-j(\boldsymbol{k}\cdot\boldsymbol{r})} \boldsymbol{a}_H = \frac{E_{so}}{\eta} e^{-j(k_x x + k_y y + k_z z)} \boldsymbol{a}_H$$

Note that the components of $\boldsymbol{k}$ define the plane wave phase shift (and attenuation, in the case of a lossy medium) in the component directions. The unit vectors $\boldsymbol{a}_E$ and $\boldsymbol{a}_H$ are located in the plane perpendicular to the direction of propagation defined by $\boldsymbol{a}_k$.

Obliquely Incident Plane Waves

Any plane wave which is obliquely incident on a planar media interface can be represented by a linear combination of two special cases: *parallel polarization* and *perpendicular polarization*. In order to define these polarization geometries, we must first define the *plane of incidence*.

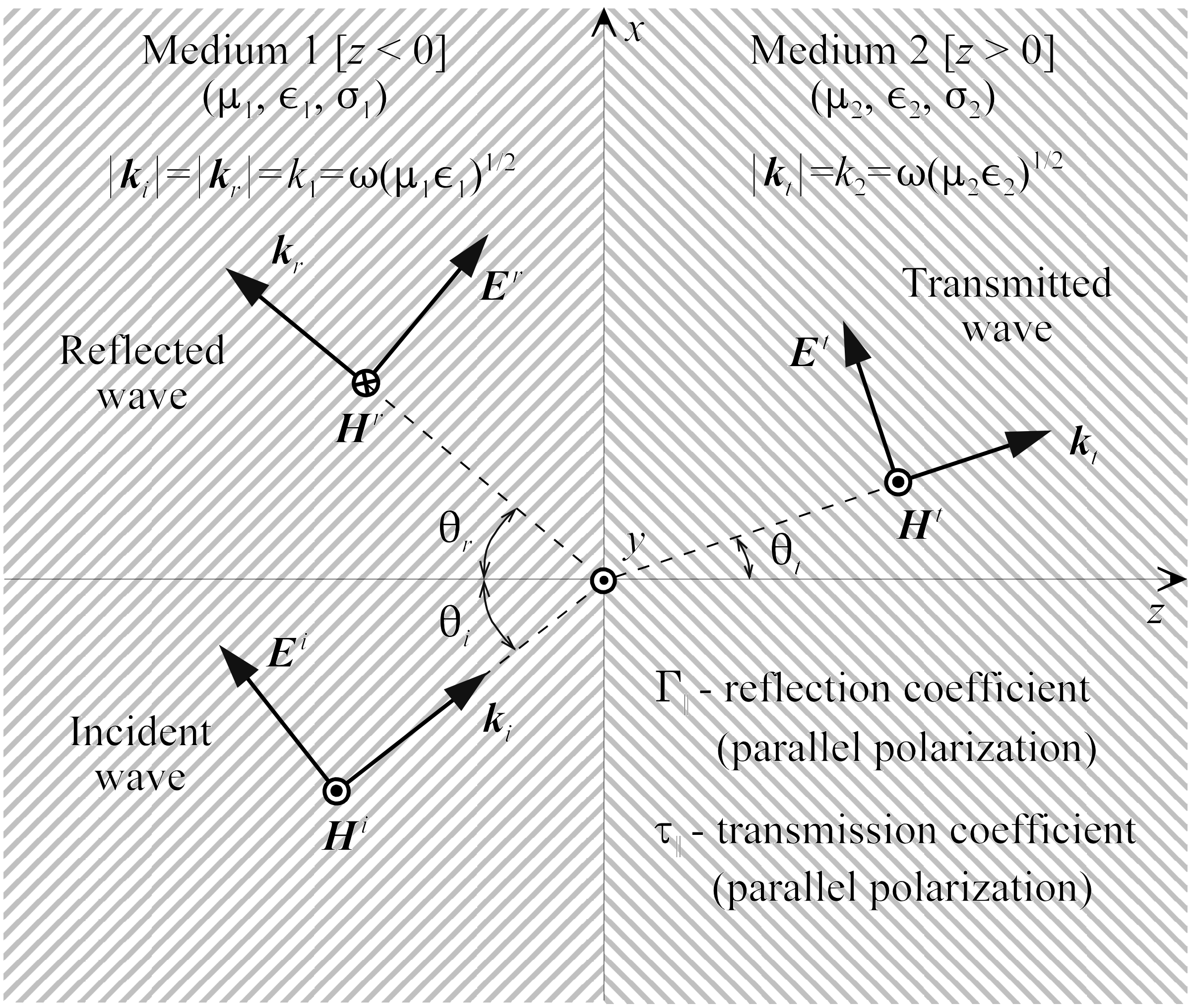
Plane of incidence - the plane containing the propagation vector of the incident wave $\boldsymbol{k}_i$ and the unit normal to the interface.

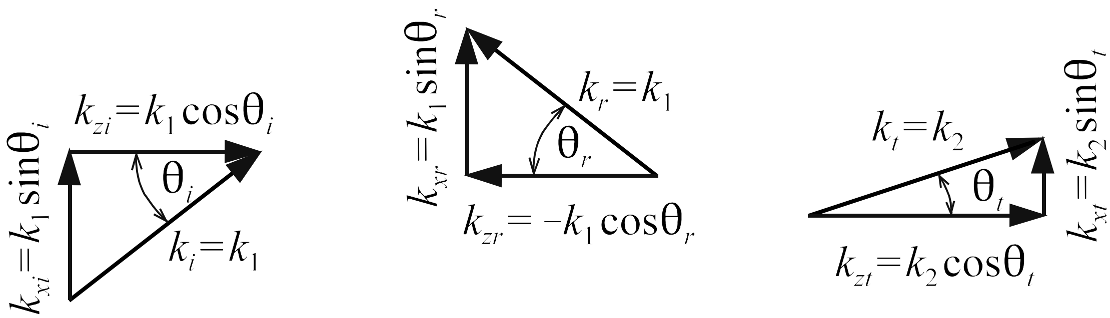
Parallel polarization - the electric field of the incident wave lies in the plane of incidence.

Perpendicular polarization - the electric field of the incident wave lies normal to the plane of incidence.

Parallel Polarization

For the media interface shown below, the plane of incidence containing the propagation vector and the normal to the interface is the *x-z* plane. The electric field of the incident wave also lies in the plane of incidence such that this wave orientation is defined as *parallel polarization*.





Using the basic concepts of the reflection and transmission coefficients, the phasor electric and magnetic fields associated with the incident, reflected and transmitted waves may be written as

$$\boldsymbol{E}_s^i = E_{so}\,\boldsymbol{a}_{Ei}\, e^{-j(k_{xi}x + k_{zi}z)} = E_{so}(\cos\theta_i \boldsymbol{a}_x - \sin\theta_i \boldsymbol{a}_z)\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\boldsymbol{E}_s^r = \Gamma_\parallel E_{so}\,\boldsymbol{a}_{Er}\, e^{-j(k_{xr}x + k_{zr}z)} = \Gamma_\parallel E_{so}(\cos\theta_r \boldsymbol{a}_x + \sin\theta_r \boldsymbol{a}_z)\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\boldsymbol{E}_s^t = \tau_\parallel E_{so}\,\boldsymbol{a}_{Et}\, e^{-j(k_{xt}x + k_{zt}z)} = \tau_\parallel E_{so}(\cos\theta_t \boldsymbol{a}_x - \sin\theta_t \boldsymbol{a}_z)\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\boldsymbol{H}_s^i = \frac{E_{so}}{\eta_1}\boldsymbol{a}_y\, e^{-j(k_{xi}x + k_{zi}z)} = \frac{E_{so}}{\eta_1}\boldsymbol{a}_y\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\boldsymbol{H}_s^r = \Gamma_\parallel \frac{E_{so}}{\eta_1}(-\boldsymbol{a}_y)\, e^{-j(k_{xr}x + k_{zr}z)} = -\Gamma_\parallel \frac{E_{so}}{\eta_1}\boldsymbol{a}_y\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\boldsymbol{H}_s^t = \tau_\parallel \frac{E_{so}}{\eta_2}\boldsymbol{a}_y\, e^{-j(k_{xt}x + k_{zt}z)} = \tau_\parallel \frac{E_{so}}{\eta_2}\boldsymbol{a}_y\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$$

The reflection and transmission coefficients are found by enforcing the boundary conditions on the tangential electric and magnetic fields at the interface ($z = 0$).

$$E_{sx}^i + E_{sx}^r = E_{sx}^t \qquad \text{at } z = 0$$

$$H_{sy}^i + H_{sy}^r = H_{sy}^t \qquad \text{at } z = 0$$

The resulting equations are

$$\cos\theta_i\, e^{-jk_1(x\sin\theta_i)} + \Gamma_\parallel \cos\theta_r\, e^{-jk_1(x\sin\theta_r)} = \tau_\parallel \cos\theta_t\, e^{-jk_2(x\sin\theta_t)} \qquad ①$$

$$\frac{1}{\eta_1} e^{-jk_1(x\sin\theta_i)} - \frac{\Gamma_\parallel}{\eta_1} e^{-jk_1(x\sin\theta_r)} = \frac{\tau_\parallel}{\eta_2} e^{-jk_2(x\sin\theta_t)} \qquad ②$$

Equations ① and ② must be valid for any value of x on the interface. This requires that the exponential terms found in ① and ② must be equal.

Equating the complex exponential terms found in ① and ② yields

$$k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t$$

or

$$\theta_i = \theta_r \qquad (\text{incident angle} = \text{reflected angle})$$

$$\frac{k_1}{k_2} = \frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\mu_1 \epsilon_1}{\mu_2 \epsilon_2}} = \frac{n_1}{n_2} \qquad (\text{Snell's law})$$

where n_1 and n_2 are the *refractive indices* of the two media. The refractive indices may be written as

$$n_1 = c\sqrt{\mu_1 \epsilon_1} = \frac{c}{u_1} \qquad n_2 = c\sqrt{\mu_2 \epsilon_2} = \frac{c}{u_2}$$

If the media properties and angle of incidence are known, we may use Snell's law to determine the angle of transmission θ_t. Given that all of the complex exponential terms in ① and ② are equal, these equations reduce to

$$(1 + \Gamma_\parallel)\cos\theta_i = \tau_\parallel \cos\theta_t \qquad ③$$

$$\frac{1 - \Gamma_\parallel}{\eta_1} = \frac{\tau_\parallel}{\eta_2} \qquad ④$$

Solving ③ and ④ for $\Gamma_\parallel$ and $\tau_\parallel$ yields

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \qquad ⑤$$

$$\tau_\parallel = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \qquad ⑥$$

Equations ⑤ and ⑥ are commonly referred to as the *Fresnel equations* for parallel polarization. These equations reduce to the normal incidence equations when θ_i=0.

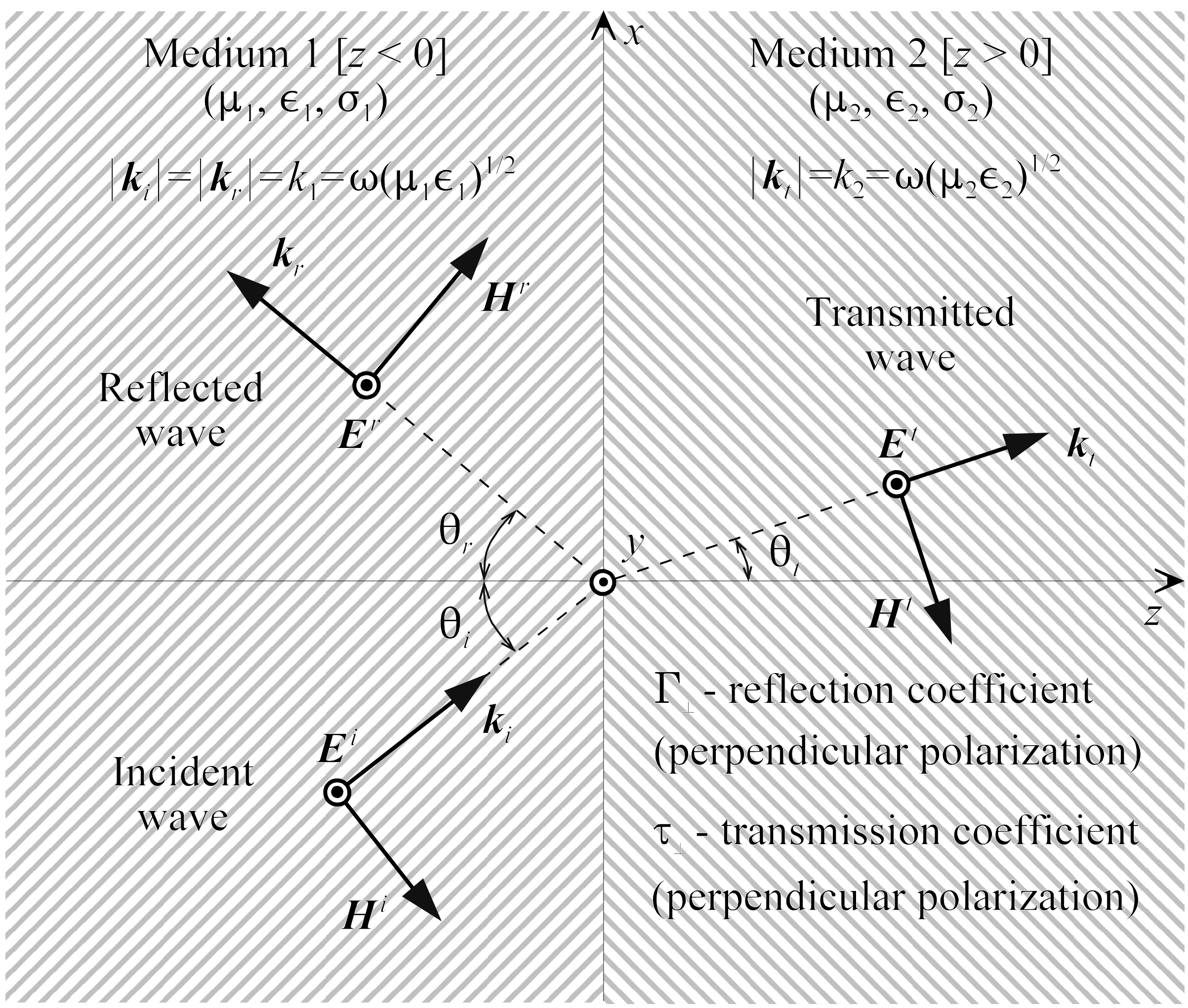
According to equation ⑤, it is possible to achieve total transmission ($\Gamma_{\parallel}=0$) at angle which is dependent on the properties of the two media. This angle is called the *Brewster angle* ($\theta_{B\parallel}$) which for two lossless dielectrics is defined by

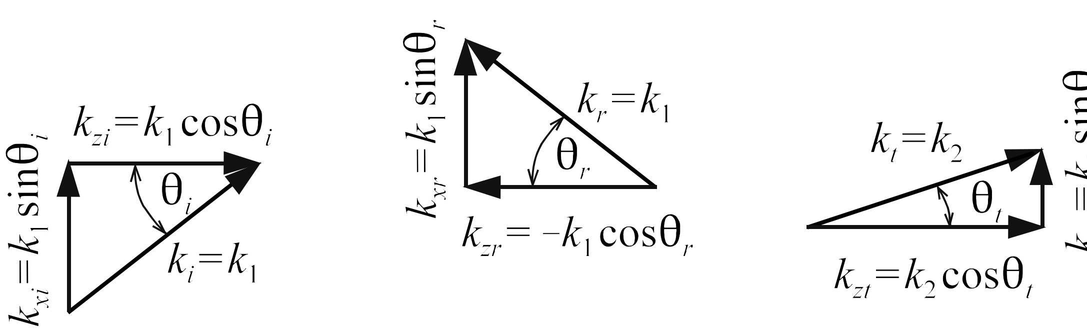
$$\theta_{B\parallel} = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1 + \epsilon_2}}$$

The Brewster angle can be used in the design of optical lenses to transmit specific polarizations. The output of a laser source (random polarization) can be passed through a glass plate positioned at the Brewster angle to provide a polarized output.

Perpendicular Polarization

For perpendicular polarization, the electric field of the incident wave is perpendicular to the plane of incidence (*x*-*z* plane for the example below).





The electric and magnetic fields of the incident, reflected and transmitted waves in the case of perpendicular polarization are

$$\boldsymbol{E}_s^i = E_{so}\boldsymbol{a}_y e^{-j(k_{xi}x + k_{zi}z)} = E_{so}\boldsymbol{a}_y\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$$
$$\boldsymbol{E}_s^r = \Gamma_\perp E_{so}\boldsymbol{a}_y e^{-j(k_{xr}x + k_{zr}z)} = \Gamma_\perp E_{so}\boldsymbol{a}_y\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$$
$$\boldsymbol{E}_s^t = \tau_\perp E_{so}\boldsymbol{a}_y e^{-j(k_{xt}x + k_{zt}z)} = \tau_\perp E_{so}\boldsymbol{a}_y\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$$
$$\boldsymbol{H}_s^i = \frac{E_{so}}{\eta_1}\boldsymbol{a}_{Hi}\, e^{-j(k_{xi}x + k_{zi}z)} = \frac{E_{so}}{\eta_1}(-\cos\theta_i\boldsymbol{a}_x + \sin\theta_i\boldsymbol{a}_z)\, e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}$$
$$\boldsymbol{H}_s^r = \Gamma_\perp\frac{E_{so}}{\eta_1}\boldsymbol{a}_{Hr}\, e^{-j(k_{xr}x + k_{zr}z)} = \Gamma_\perp\frac{E_{so}}{\eta_1}(\cos\theta_r\boldsymbol{a}_x + \sin\theta_r\boldsymbol{a}_z)\, e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}$$
$$\boldsymbol{H}_s^t = \tau_\perp\frac{E_{so}}{\eta_2}\boldsymbol{a}_{Ht}\, e^{-j(k_{xt}x + k_{zt}z)} = \tau_\perp\frac{E_{so}}{\eta_2}(-\cos\theta_t\boldsymbol{a}_x + \sin\theta_t\boldsymbol{a}_z)\, e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}$$

Enforcement of the boundary conditions and solving for the reflection and transmission coefficients yields the Fresnel equations for perpendicular polarization.

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$
$$\tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$