Antenna Patterns

(Radiation Patterns)

Antenna Pattern - a graphical representation of the antenna radiation properties as a function of position (spherical coordinates).

Common Types of Antenna Patterns

Power Pattern - normalized power vs. spherical coordinate position.

Field Pattern - normalized $|\boldsymbol{E}|$ or $|\boldsymbol{H}|$ vs. spherical coordinate position.

Antenna Field Types

Reactive field - the portion of the antenna field characterized by standing (stationary) waves which represent stored energy.

Radiation field - the portion of the antenna field characterized by radiating (propagating) waves which represent transmitted energy.

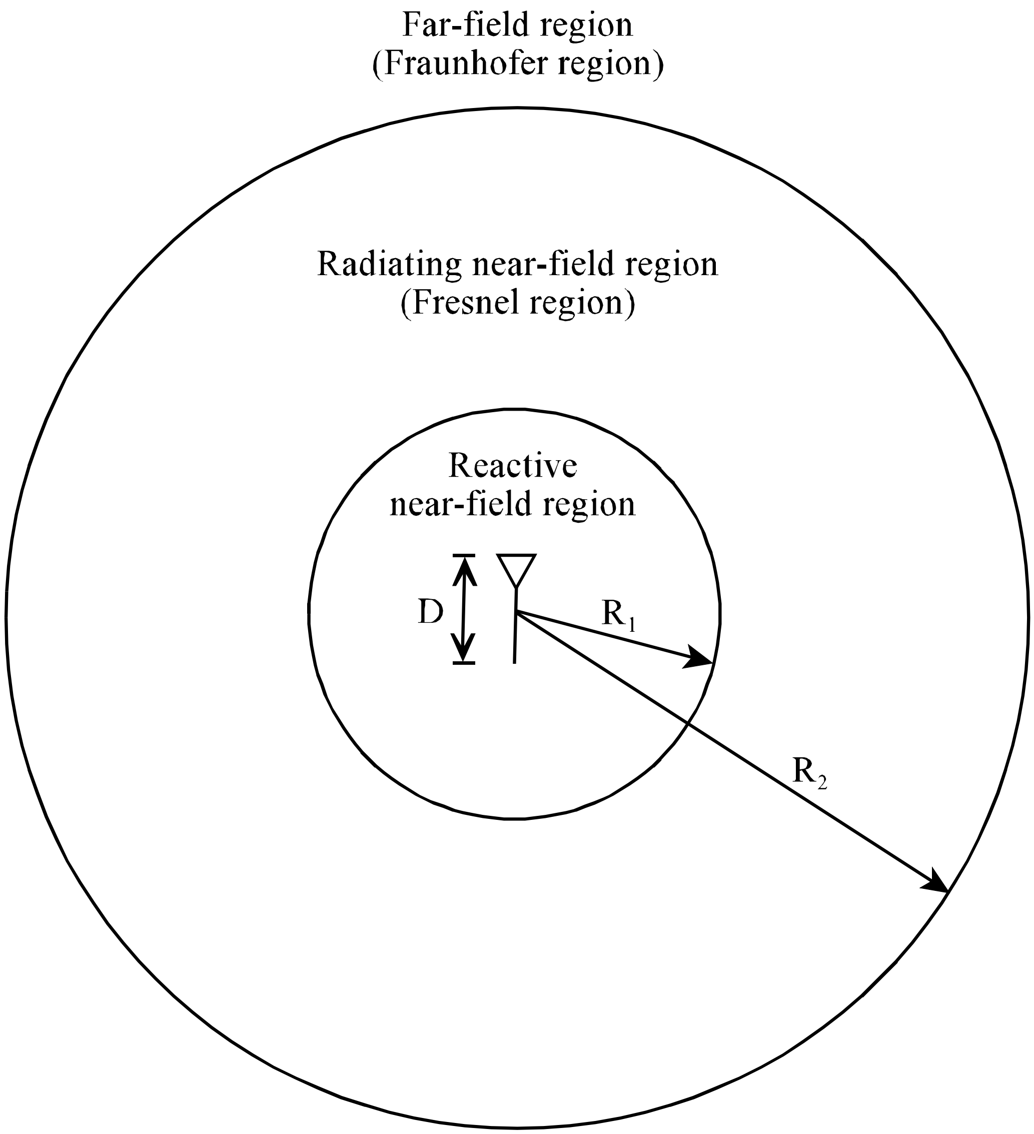
Antenna Field Regions

Reactive Near Field Region - the region immediately surrounding the antenna where the reactive field (stored energy - standing waves) is dominant.

Near-Field (Fresnel) Region - the region between the reactive near-field and the far-field where the radiation fields are dominant and the field distribution is dependent on the distance from the antenna.

Far-Field (Fraunhofer) Region - the region farthest away from the antenna where the field distribution is essentially independent of the distance from the antenna (propagating waves).

Antenna Field Regions



D = maximum antenna dimension

$$R_1 = 0.62\sqrt{\frac{D^3}{\lambda}}$$

$$R_2 = \frac{2D^2}{\lambda}$$

Antenna Pattern Definitions

Isotropic Pattern - an antenna pattern defined by uniform radiation in all directions, produced by an isotropic radiator (point source, a non-physical antenna which is the only nondirectional antenna).

Directional Pattern - a pattern characterized by more efficient radiation in one direction than another (all physically realizable antennas are directional antennas).

Omnidirectional Pattern - a pattern which is uniform in a given plane.

Principal Plane Patterns - the E-plane and H-plane patterns of a linearly polarized antenna.

E-plane - the plane containing the electric field vector and the direction of maximum radiation.

H-plane - the plane containing the magnetic field vector and the direction of maximum radiation.

Antenna Pattern Parameters

Radiation Lobe - a clear peak in the radiation intensity surrounded by regions of weaker radiation intensity.

Main Lobe (major lobe, main beam) - radiation lobe in the direction of maximum radiation.

Minor Lobe - any radiation lobe other than the main lobe.

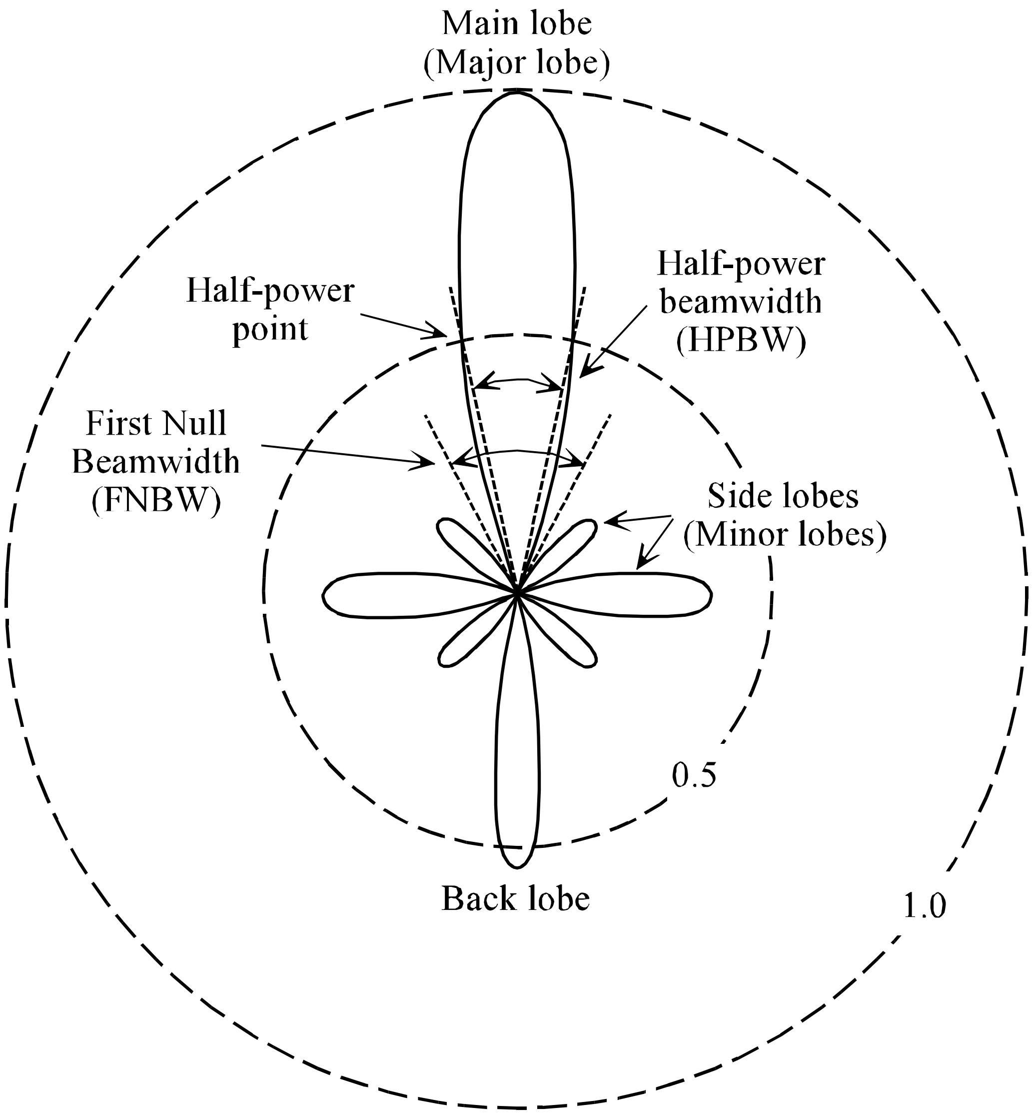
Side Lobe - a radiation lobe in any direction other than the direction(s) of intended radiation.

Back Lobe - the radiation lobe opposite to the main lobe.

Half-Power Beamwidth (HPBW) - the angular width of the main beam at the half-power points.

First Null Beamwidth (FNBW) - angular width between the first nulls on either side of the main beam.

Antenna Pattern Parameters
(Normalized Power Pattern)



Maxwell's Equations
(Instantaneous and Phasor Forms)

Maxwell's Equations (instantaneous form)

$$\nabla \times \mathscr{E} = -\frac{\partial \mathscr{B}}{\partial t}$$

$$\nabla \times \mathscr{H} = \frac{\partial \mathscr{D}}{\partial t} + \mathscr{J}$$

$$\nabla \cdot \mathscr{D} = \rho_t$$

$$\nabla \cdot \mathscr{B} = 0$$

$\mathscr{E}, \mathscr{H}, \mathscr{D}, \mathscr{B}, \mathscr{J}$ - instantaneous vectors [$\mathscr{E} = \mathscr{E}$(x,y,z,t), etc.]
ρ_t - instantaneous scalar

Maxwell's Equations (phasor form, time-harmonic form)

$$\nabla \times \boldsymbol{E} = -j\omega \boldsymbol{B}$$

$$\nabla \times \boldsymbol{H} = j\omega \boldsymbol{D} + \boldsymbol{J}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

E, ***H***, ***D***, ***B***, ***J*** - phasor vectors [***E***=***E***(x,y,z), etc.]
ρ - phasor scalar

Relation of instantaneous quantities to phasor quantities ...

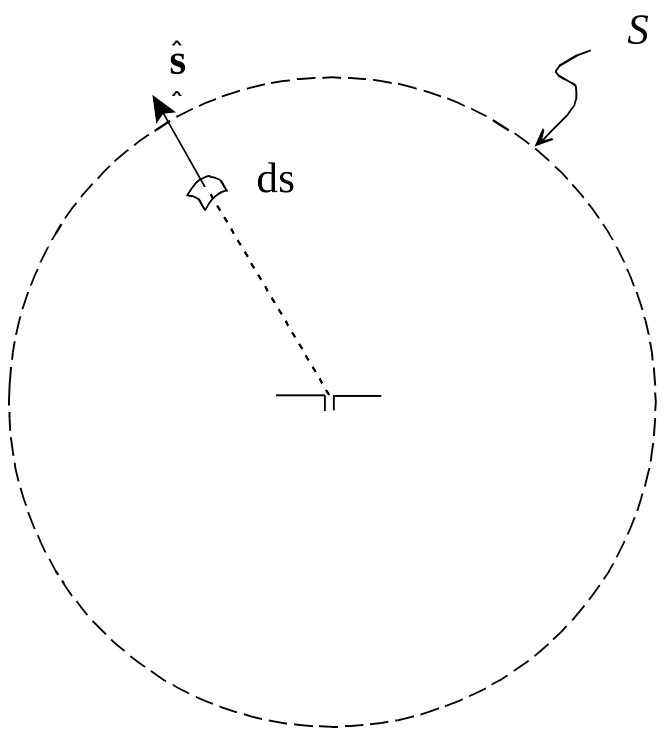
$\mathscr{E}$(x,y,z,t) = Re{***E***(x,y,z)$e^{j\omega t}$}, etc.

Average Power Radiated by an Antenna

To determine the average power radiated by an antenna, we start with the instantaneous Poynting vector $\mathscr{S}$ (vector power density) defined by

$$\mathscr{S} = \mathscr{E} \times \mathscr{H} \qquad (\text{V/m} \times \text{A/m} = \text{W/m}^2)$$

Assume the antenna is enclosed by some surface *S*.



The total instantaneous radiated power $\mathscr{P}_{rad}$ leaving the surface *S* is found by integrating the instantaneous Poynting vector over the surface.

$$\mathscr{P}_{rad} = \oint_S \mathscr{S} \cdot d\mathbf{s} = \oint_S (\mathscr{E} \times \mathscr{H}) \cdot d\mathbf{s} \qquad d\mathbf{s} = \hat{\mathbf{s}}\, ds$$

ds = differential surface
$\hat{\mathbf{s}}$ = unit vector normal to ds

For time-harmonic fields, the time average instantaneous Poynting vector (time average vector power density) is found by integrating the instantaneous Poynting vector over one period (T) and dividing by the period.

$$\boldsymbol{P}_{avg} = \frac{1}{T}\oint_T (\mathcal{E} \times \mathcal{H})\, dt$$

$$\mathcal{E} = \mathrm{Re}\{\boldsymbol{E} e^{j\omega t}\}$$

$$\mathcal{H} = \mathrm{Re}\{\boldsymbol{H} e^{j\omega t}\}$$

The instantaneous magnetic field may be rewritten as

$$\mathcal{H} = \mathrm{Re}\{\tfrac{1}{2}\, [\, \boldsymbol{H} e^{j\omega t} + \boldsymbol{H}^{*} e^{-j\omega t}\,]\}$$

which gives an instantaneous Poynting vector of

$$\mathcal{E} \times \mathcal{H} = \tfrac{1}{2}\, \mathrm{Re}\, \{\underbrace{[\boldsymbol{E} \times \boldsymbol{H}] e^{j2\omega t}}_{\substack{\text{time-harmonic} \\ \text{(integrates to zero over } T\text{)}}} + \underbrace{[\boldsymbol{E} \times \boldsymbol{H}^{*}]}_{\text{independent of time}}\}$$

and the time-average vector power density becomes

$$\boldsymbol{P}_{avg} = \frac{1}{2T} \mathrm{Re}\, [\boldsymbol{E} \times \boldsymbol{H}^{*}] \oint_T dt$$

$$= \tfrac{1}{2}\, \mathrm{Re}\, [\boldsymbol{E} \times \boldsymbol{H}^{*}]$$

The total time-average power radiated by the antenna (P_{rad}) is found by integrating the time-average power density over S.

$$P_{rad} = \oint_S \boldsymbol{P}_{avg} \cdot d\mathbf{s} = \tfrac{1}{2}\, \mathrm{Re} \oint_S [\boldsymbol{E} \times \boldsymbol{H}^{*}] \cdot d\mathbf{s}$$

Radiation Intensity

Radiation Intensity - radiated power per solid angle (radiated power normalized to a unit sphere).

$$P_{rad} = \oint_S \boldsymbol{P}_{avg} \cdot d\mathbf{s}$$

In the far field, the radiation electric and magnetic fields vary as $1/r$ and the direction of the vector power density ($\boldsymbol{P}_{avg}$) is radially outward. If we assume that the surface S is a sphere of radius r, then the integral for the total time-average radiated power becomes

$$\boldsymbol{P}_{avg} = P_{avg}\,\hat{\boldsymbol{r}}$$

$$d\boldsymbol{s} = \hat{\boldsymbol{s}}\,ds = \hat{\boldsymbol{r}}\,r^2 \sin\theta\, d\theta\, d\phi$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} P_{avg}\, r^2 \sin\theta\, d\theta\, d\phi$$

If we defined $P_{avg} r^2 = U(\theta,\phi)$ as the radiation intensity, then

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\Omega$$

where $d\Omega = \sin\theta d\theta d\phi$ defines the differential solid angle. The units on the radiation intensity are defined as watts per unit solid angle. The average radiation intensity is found by dividing the radiation intensity by the area of the unit sphere (4π) which gives

$$U_{avg} = \frac{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\, d\Omega}{4\pi} = \frac{P_{rad}}{4\pi}$$

The average radiation intensity for a given antenna represents the radiation intensity of a point source producing the same amount of radiated power as the antenna.

Directivity

Directivity (*D*) - the ratio of the radiation intensity in a given direction from the antenna to the radiation intensity averaged over all directions.

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_{avg}} = \frac{4\pi\, U(\theta,\phi)}{P_{rad}}$$

The directivity of an isotropic radiator is $D(\theta,\phi) = 1$.

The maximum directivity is defined as $[D(\theta,\phi)]_{max} = D_o$.

The directivity range for any antenna is $0 \leq D(\theta,\phi) \leq D_o$.

Directivity in dB

$$D(\theta,\phi)\ [dB] = 10\log_{10} D(\theta,\phi)$$

Directivity in terms of Beam Solid Angle

We may define the radiation intensity as

$$U(\theta,\phi) = B_o\, F(\theta,\phi)$$

where B_o is a constant and $F(\theta,\phi)$ is the radiation intensity pattern function. The directivity then becomes

$$D(\theta,\phi) = 4\pi \frac{U(\theta,\phi)}{P_{rad}} = 4\pi\, B_o \frac{F(\theta,\phi)}{P_{rad}}$$

and the radiated power is

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = B_o \int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

Inserting the expression for P_{rad} into the directivity expression yields

$$D(\theta,\phi) = 4\pi \frac{F(\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta \, d\theta \, d\phi}$$

The maximum directivity is

$$D_o = [D(\theta,\phi)]_{\max} = 4\pi \frac{[F(\theta,\phi)]_{\max}}{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta \, d\theta \, d\phi} = \frac{4\pi}{\Omega_A}$$

where the term Ω_A in the previous equation is defined as the *beam solid angle* and is defined by

$$\Omega_A = \frac{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta \, d\theta \, d\phi}{[F(\theta,\phi)]_{\max}} = \int_0^{2\pi}\int_0^{\pi} F_n(\theta,\phi)\sin\theta \, d\theta \, d\phi$$

$$F_n(\theta,\phi) = \frac{F(\theta,\phi)}{[F(\theta,\phi)]_{\max}}$$

Beam Solid Angle - the solid angle through which all of the antenna power would flow if the radiation intensity were $[U(\theta,\phi)]_{\max}$ for all angles in Ω_A.

<u>Example</u> (Directivity/Beam Solid Angle/Maximum Directivity)

Determine the directivity $[D(\theta,\phi)]$, the beam solid angle Ω_A and the maximum directivity $[D_o]$ of an antenna defined by $F(\theta,\phi) = \sin^2\theta\cos^2\theta$.

$$D(\theta,\phi) = 4\pi \frac{F(\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi}$$

$$= 4\pi \frac{\sin^2\theta\cos^2\theta}{\int_0^{2\pi}\int_0^{\pi} \sin^3\theta\cos^2\theta\, d\theta\, d\phi}$$

$$\sin^3\theta\cos^2\theta = \sin^3\theta(1 - \sin^2\theta) = \sin^3\theta - \sin^5\theta$$

$$D(\theta,\phi) = 4\pi \frac{\sin^2\theta\cos^2\theta}{2\pi\int_0^{\pi} (\sin^3\theta - \sin^5\theta)\, d\theta}$$

$$\int \sin^3 x\, dx = -\frac{1}{3}(\cos x)(\sin^2 x + 2)$$

$$\int \sin^5 x\, dx = -\frac{\sin^4 x\cos x}{5} - \frac{4}{15}(\cos x)(\sin^2 x + 2)$$

$$D(\theta,\phi) = 4\pi \frac{\sin^2\theta\cos^2\theta}{2\pi\left(\frac{4}{3} - \frac{16}{15}\right)} = 4\pi \frac{\sin^2\theta\cos^2\theta}{\left(\frac{8\pi}{15}\right)}$$

$$D(\theta,\phi) = \frac{15}{2}\sin^2\theta\cos^2\theta$$

$$\Omega_A = \frac{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi}{[F(\theta,\phi)]_{\mathbf{max}}}$$

In order to find $[F(\theta,\phi)]_{max}$, we must solve

$$\frac{dF(\theta,\phi)}{d\theta} = \frac{d}{d\theta}\left(\sin^2\theta\cos^2\theta\right) = 0$$

$$(2\sin\theta\cos\theta)\cos^2\theta + \sin^2\theta(-2\cos\theta\sin\theta) = 0$$

$$\sin\theta\cos^3\theta - \sin^3\theta\cos\theta = 0$$

$$\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = 0$$

$$\sin\theta\cos\theta(1 - 2\sin^2\theta) = 0$$

$\sin\theta = 0$	$\theta = (0,\pi)$	(*minimums*)
$\cos\theta = 0$	$\theta = \frac{\pi}{2}$	(*minimum*)
$1 - 2\sin^2\theta = 0$	$\theta = \sin^{-1}\left(\frac{\pm 1}{\sqrt{2}}\right) = \frac{\pi}{4}, \frac{3\pi}{4}$	(*maximums*)

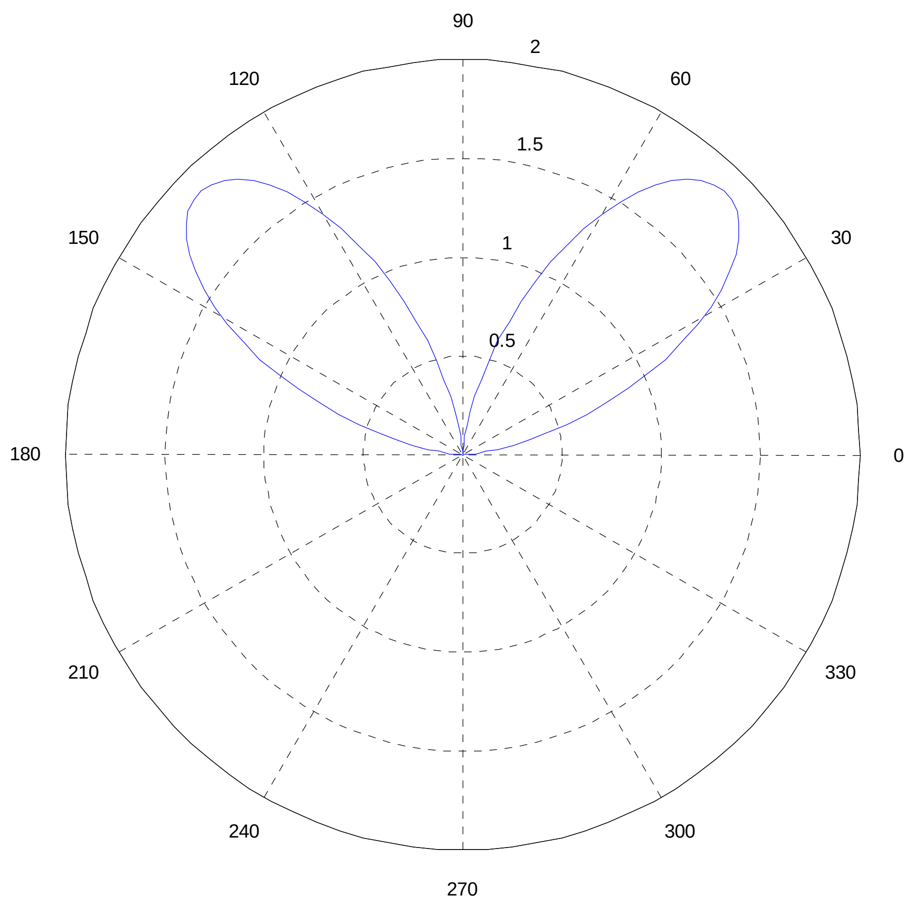
$$[F(\theta,\phi)]_{\mathbf{max}} = \sin^2\left(\frac{\pi}{4}\right)\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{4}$$

$$\Omega_A = \frac{(8\pi/15)}{(1/4)} = \frac{32\pi}{15} \ \text{rad}^2 = 6.70 \ \text{rad}^2$$

$$D_o = \frac{4\pi}{\Omega_A} = 4\pi \frac{(15)}{(32\pi)} = \frac{15}{8} = 1.875 \ (2.73 \ \text{dB})$$

MATLAB m-file for plotting this directivity function

```
for i=1:100
   theta(i)=pi*(i-1)/99;
   d(i)=7.5*((cos(theta(i)))^2)*((sin(theta(i)))^2);
end
polar(theta,d)
```



Directivity/Beam Solid Angle Approximations

Given an antenna with one *narrow* major lobe and negligible radiation in its minor lobes, the beam solid angle may be approximated by

$$\Omega_A \approx \theta_1 \theta_2$$

where θ_1 and θ_2 are the half-power beamwidths (in radians) which are perpendicular to each other. The maximum directivity, in this case, is approximated by

$$D_o = \frac{4\pi}{\Omega_A} \approx \frac{4\pi}{\theta_1\theta_2} \qquad (\theta_1, \theta_2 \text{ in radians})$$

If the beamwidths are measured in degrees, we have

$$D_o \approx \frac{4\pi(180/\pi)^2}{\theta_1\theta_2} = \frac{41{,}253}{\theta_1\theta_2} \qquad (\theta_1, \theta_2 \text{ in degrees})$$

Example (Approximate Directivity)

A horn antenna with low side lobes has half-power beamwidths of 29° in both principal planes (*E*-plane and *H*-plane). Determine the approximate directivity (dB) of the horn antenna.

$$D_o \approx \frac{41{,}253}{29^2} = 49.05$$

$$D_o(\text{dB}) = 10\log_{10}(49.05) = 16.9 \text{ dB}$$

Numerical Evaluation of Directivity

The maximum directivity of a given antenna may be written as

$$D_o = 4\pi \frac{[U(\theta,\phi)]_{max}}{P_{rad}}$$

$$= 4\pi \frac{[U(\theta,\phi)]_{max}}{\int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi}$$

$$= 4\pi \frac{[F(\theta,\phi)]_{max}}{\int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi}$$

where $U(\theta\phi) = B_o F(\theta,\phi)$. The integrals related to the radiated power in the denominators of the terms above may not be analytically integrable. In this case, the integrals must be evaluated using numerical techniques. If we assume that the dependence of the radiation intensity on θ and ϕ is separable, then we may write

$$U(\theta,\phi) = B_o F(\theta,\phi) = B_o f(\theta) g(\phi)$$

The radiated power integral then becomes

$$P_{rad} = B_o \int_0^{2\pi}\int_0^{\pi} f(\theta) g(\phi) \sin\theta\, d\theta\, d\phi$$

$$= B_o \left[\int_0^{\pi} f(\theta)\sin\theta\, d\theta\right]\left[\int_0^{2\pi} g(\phi)\, d\phi\right]$$

Note that the assumption of a separable radiation intensity pattern function results in the product of two separate integrals for the radiated power. We may employ a variety of numerical integration techniques to evaluate the integrals. The most straightforward of these techniques is the rectangular rule (others include the trapezoidal rule, Gaussian quadrature, etc.) If we first consider the θ-dependent integral, the range of θ is first subdivided into N equal intervals of length

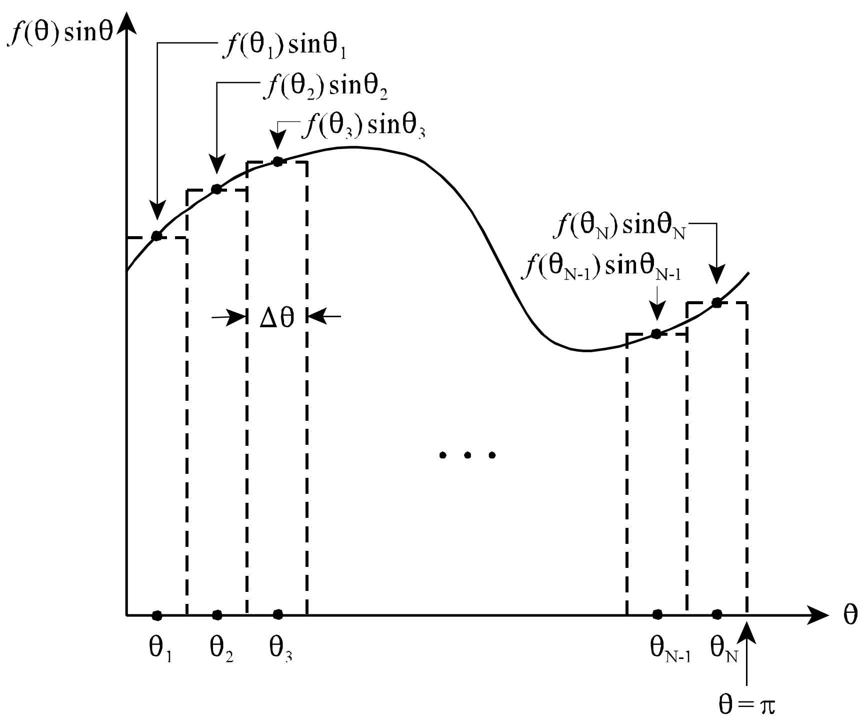
$$\Delta\theta = \frac{\pi}{N}$$

The known function $f(\theta)$ is then evaluated at the center of each subinterval. The center of each subinterval is defined by

$$\theta_i = \Delta\theta\left(i - \frac{1}{2}\right) = \frac{\pi}{N}\left(i - \frac{1}{2}\right) \qquad i=1,2,...,N$$

The area of each rectangular sub-region is given by

$$[f(\theta_i)\sin\theta_i]\Delta\theta$$



The overall integral is then approximated by

$$\int_0^{\pi} f(\theta)\sin\theta\, d\theta \approx \sum_{i=1}^{N} [f(\theta_i)\sin\theta_i]\Delta\theta = \Delta\theta \sum_{i=1}^{N} f(\theta_i)\sin\theta_i$$

Using the same technique on the ϕ-dependent integral yields

$$\Delta\phi = \frac{2\pi}{M}$$

$$\phi_i = \Delta\phi\left(j - \frac{1}{2}\right) = \frac{2\pi}{M}\left(j - \frac{1}{2}\right) \qquad j = 1,2,\ldots,M$$

$$\int_0^{2\pi} g(\phi)\, d\phi \approx \sum_{j=1}^{M} [g(\phi_j)]\Delta\phi = \Delta\phi \sum_{j=1}^{M} g(\phi_j)$$

Combining the θ and ϕ dependent integration results gives the approximate radiated power.

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = B_o \int_0^{2\pi}\int_0^{\pi} F(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$\approx B_o \Delta\theta\Delta\phi \left[\sum_{i=1}^{N} f(\theta_i)\sin\theta_i\right]\left[\sum_{j=1}^{M} g(\phi_j)\right]$$

$$= \frac{2\pi^2 B_o}{NM}\left[\sum_{i=1}^{N} f(\theta_i)\sin\theta_i\right]\left[\sum_{j=1}^{M} g(\phi_j)\right]$$

The approximate radiated power for antennas that are omnidirectional with respect to ϕ [$g(\phi) = 1$] reduces to

$$P_{rad} = 2\pi B_o \Delta\theta \left[\sum_{i=1}^{N} f(\theta_i)\sin\theta_i\right] = \frac{2\pi^2 B_o}{N}\left[\sum_{i=1}^{N} f(\theta_i)\sin\theta_i\right]$$

The approximate radiated power for antennas that are omnidirectional with respect to θ [$f(\theta) = 1$] reduces to

$$P_{rad} \approx 2B_o \Delta\phi \left[\sum_{j=1}^{M} g(\phi_j) \right] = \frac{4\pi B_o}{M} \left[\sum_{j=1}^{M} g(\phi_j) \right]$$

For antennas which have a radiation intensity which is not separable in θ and ϕ, the a two-dimensional numerical integration must be performed which yields

$$P_{rad} \approx \frac{2\pi^2 B_o}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \left[F(\theta_i, \phi_j) \sin\theta_i \right]$$

Example (Numerical evaluation of directivity)

Determine the directivity of a half-wave dipole given the radiation intensity of

$$U(\theta,\phi) = B_o \left[\frac{\cos\left(\frac{\pi}{2} \cos\theta \right)}{\sin\theta} \right]^2 = B_o f(\theta) \qquad [g(\phi) = 1]$$

$$D_o = 4\pi \frac{[U(\theta,\phi)]_{\max}}{P_{rad}}$$

$$P_{rad} \approx \frac{2\pi^2 B_o}{N} \sum_{i=1}^{N} \left[f(\theta_i) \sin\theta_i \right]$$

$$\theta_i = \frac{\pi}{N} \left(i - \frac{1}{2} \right) \qquad i = 1, 2, \ldots, N$$

The maximum value of the radiation intensity for a half-wave dipole occurs at $\theta = \pi/2$ so that

$$\left[U(\theta,\phi)\right]_{\max} = B_o \left[\frac{\cos\left(\frac{\pi}{2}\cos\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}} \right]^2 = B_o$$

$$D_o \approx \frac{4\pi B_o}{\frac{2\pi^2 B_o}{N}\sum_{i=1}^{N} f(\theta_i)\sin\theta_i} = \frac{\frac{2N}{\pi}}{\sum_{i=1}^{N}\left\{ \frac{\cos^2\left(\frac{\pi}{2}\cos\theta_i\right)}{\sin\theta_i} \right\}}$$

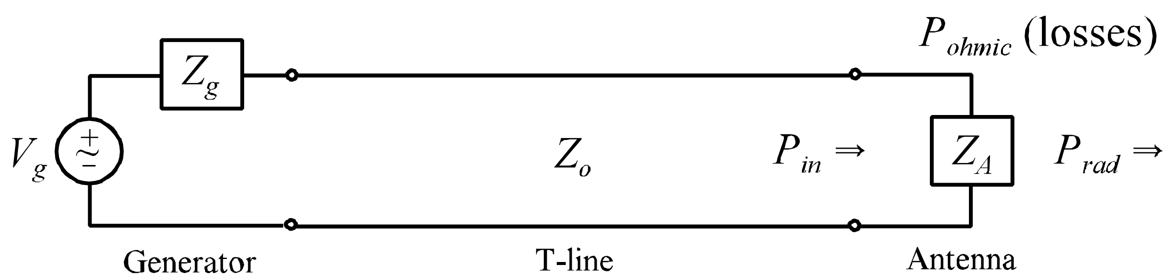
MATLAB m-file

```
sum=0.0;
N=input('Enter the number of segments in the theta direction')
for i=1:N
   thetai=(pi/N)*(i-0.5);
   sum=sum+(cos((pi/2)*cos(thetai)))^2/sin(thetai);
end
D=(2*N)/(pi*sum)
```

N	D_o
5	1.6428
10	1.6410
15	1.6409
20	1.6409

Antenna Efficiency

When an antenna is driven by a voltage source (generator), the total power radiated by the antenna will not be the total power available from the generator. The loss factors which affect the antenna efficiency can be identified by considering the common example of a generator connected to a transmitting antenna via a transmission line as shown below.



Z_g - source impedance

Z_A - antenna impedance

Z_o - transmission line characteristic impedance

P_{in} - total power delivered to the antenna terminals

P_{ohmic} - antenna ohmic (I^2R) losses
[conduction loss + dielectric loss]

P_{rad} - total power radiated by the antenna

The total power delivered to the antenna terminals is less than that available from the generator given the effects of mismatch at the source/t-line connection, losses in the t-line, and mismatch at the t-line/antenna connection. The total power delivered to the antenna terminals must equal that lost to I^2R (ohmic) losses plus that radiated by the antenna.

$$P_{in} = P_{rad} + P_{ohmic}$$

We may define the *antenna radiation efficiency* (e_{cd}) as

$$e_{cd} = \frac{P_{rad}}{P_{in}} = \frac{P_{rad}}{P_{rad} + P_{ohmic}}$$

which gives a measure of how efficient the antenna is at radiating the power delivered to its terminals. The antenna radiation efficiency may be written as a product of the conduction efficiency (e_c) and the dielectric efficiency (e_d).

$$e_{cd} = e_c e_d$$

e_c - conduction efficiency (conduction losses only)

e_d - dielectric efficiency (dielectric losses only)

However, these individual efficiency terms are difficult to compute so that they are typically determined by experimental measurement. This antenna measurement yields the total antenna radiation efficiency such that the individual terms cannot be separated.

Note that the antenna radiation efficiency does not include the mismatch (reflection) losses at the t-line/antenna connection. This loss factor is not included in the antenna radiation efficiency because it is not inherent to the antenna alone. The reflection loss factor depends on the t-line connected to the antenna. We can define the *total antenna efficiency* (e_o), which includes the losses due to mismatch as

$$e_o = e_r e_c e_d$$

e_o - total antenna efficiency (all losses)

e_r - reflection efficiency (mismatch losses)

The reflection efficiency represents the ratio of power delivered to the antenna terminals to the total power incident on the t-line/antenna connection. The reflection efficiency is easily found from transmission line theory in terms of the reflection coefficient (Γ).

$$e_r = 1 - |\Gamma|^2$$

$$\Gamma = \frac{Z_A - Z_o}{Z_A + Z_o}$$

The total antenna efficiency then becomes

$$e_o = e_{cd}(1 - |\Gamma|^2)$$

The definition of antenna efficiency (specifically, the antenna radiation efficiency) plays an important role in the definition of antenna gain.

Antenna Gain

The definitions of antenna *directivity* and antenna *gain* are essentially the same except for the power terms used in the definitions.

Directivity [$D(\theta,\phi)$] - ratio of the antenna radiated power density at a distant point to the *total antenna radiated power* (P_{rad}) radiated isotropically.

Gain [$G(\theta,\phi)$] - ratio of the antenna radiated power density at a distant point to the *total antenna input power* (P_{in}) radiated isotropically.

Thus, the antenna gain, being dependent on the total power delivered to the antenna input terminals, accounts for the ohmic losses in the antenna while the antenna directivity, being dependent on the total radiated power, does not include the effect of ohmic losses.

The equations for directivity and gain are

$$D(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{rad}} = \frac{U(\theta,\phi)}{\frac{P_{rad}}{4\pi}}$$

$$G(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{in}} = \frac{U(\theta,\phi)}{\frac{P_{in}}{4\pi}}$$

The relationship between the directivity and gain of an antenna may be found using the definition of the radiation efficiency of the antenna.

$$P_{rad} = e_{cd} P_{in}$$

$$D(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{rad}} = \frac{4\pi U(\theta,\phi)}{e_{cd} P_{in}} = \frac{G(\theta,\phi)}{e_{cd}}$$

$$G(\theta,\phi) = e_{cd} D(\theta,\phi)$$

Gain in dB

$$G(\theta,\phi)\ [dB] = 10 \log_{10} G(\theta,\phi)$$

Antenna Impedance

The complex antenna impedance is defined in terms of resistive (real) and reactive (imaginary) components.

$$Z_A = R_A + jX_A$$

R_A - Antenna resistance
[(dissipation) ohmic losses + radiation]

X_A - Antenna reactance
[(energy storage) antenna near field]

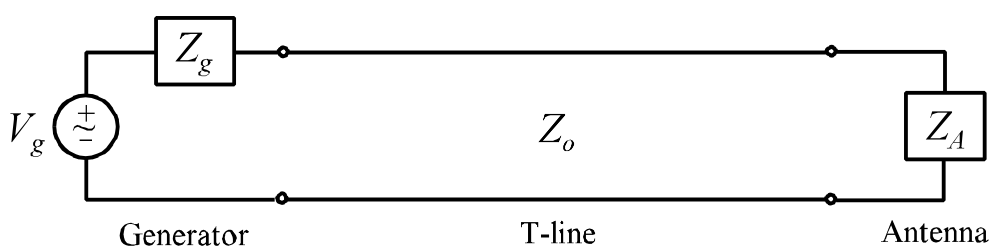
We may define the antenna resistance as the sum of two resistances which separately represent the ohmic losses and the radiation.

$$R_A = R_r + R_L$$

R_r - Antenna radiation resistance (radiation)

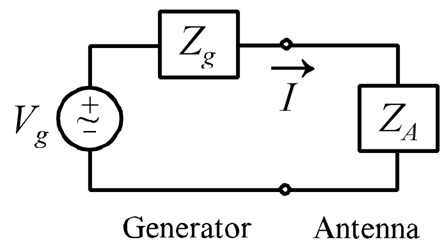
R_L - Antenna loss resistance (ohmic loss)

The typical transmitting system can be defined by a generator, transmission line and transmitting antenna as shown below.

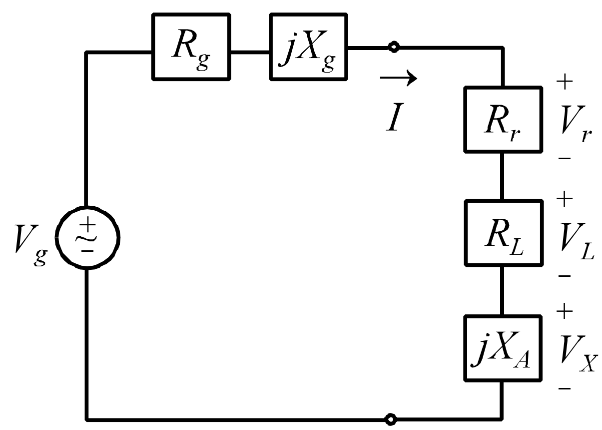


The generator is modeled by a complex source voltage V_g and a complex source impedance Z_g.

In some cases, the generator may be connected directly to the antenna.



Inserting the complete source and antenna impedances yields



The complex power associated with any element in the equivalent circuit is given by

$$P = V_{rms} I_{rms}^{*} = \frac{1}{2} V_{peak} I_{peak}^{*}$$

where the * denotes the complex conjugate. We will assume peak values for all voltages and currents in expressing the radiated power, the power associated with ohmic losses, and the reactive power in terms of specific components of the antenna impedance. The peak current for the simple series circuit shown above is

$$I = \frac{V_g}{Z_g + Z_A} = \frac{V_g}{(R_g + R_r + R_L) + j(X_g + X_A)}$$

The power radiated by the antenna (P_r) may be written as

$$P_r = \frac{1}{2} V_r I^* = \frac{1}{2}(I R_r) I^* = \frac{1}{2} |I|^2 R_r$$

$$|I| = \frac{|V_g|}{\sqrt{(R_g + R_r + R_L)^2 + (X_g + X_A)^2}}$$

$$P_r = \frac{|V_g|^2 R_r}{2\,[(R_g + R_r + R_L)^2 + (X_g + X_A)^2]}$$

The power dissipated as heat (P_L) may be written

$$P_L = \frac{1}{2} V_L I^* = \frac{1}{2}(I R_L) I^* = \frac{1}{2} |I|^2 R_L$$

$$P_L = \frac{|V_g|^2 R_L}{2\,[(R_g + R_r + R_L)^2 + (X_g + X_A)^2]}$$

The reactive power (imaginary component of the complex power) stored in the antenna near field (P_X) is

$$P_X = \frac{1}{2} V_X I^* = \frac{1}{2}(j I X_A) I^* = \frac{j}{2} |I|^2 X_A$$

$$P_X = \frac{j\,|V_g|^2 X_A}{2\,[(R_g + R_r + R_L)^2 + (X_g + X_A)^2]}$$

From the equivalent circuit for the generator/antenna system, we see that maximum power transfer occurs when

$$Z_A = Z_g^*$$

$$R_A = R_r + R_L = R_g$$

$$X_A = -X_g$$

The circuit current in this case is

$$I = \frac{V_g}{Z_g + Z_A} = \frac{V_g}{2(R_r + R_L)}$$

The power radiated by the antenna is

$$P_r = \frac{1}{2}V_r I^* = \frac{1}{2}(IR_r)I^* = \frac{1}{2}|I|^2 R_r$$

$$P_r = \frac{|V_g|^2 R_r}{8(R_r + R_L)^2}$$

The power dissipated in heat is

$$P_L = \frac{1}{2}V_L I^* = \frac{1}{2}(IR_L)I^* = \frac{1}{2}|I|^2 R_L$$

$$P_L = \frac{|V_g|^2 R_L}{8(R_r + R_L)^2}$$

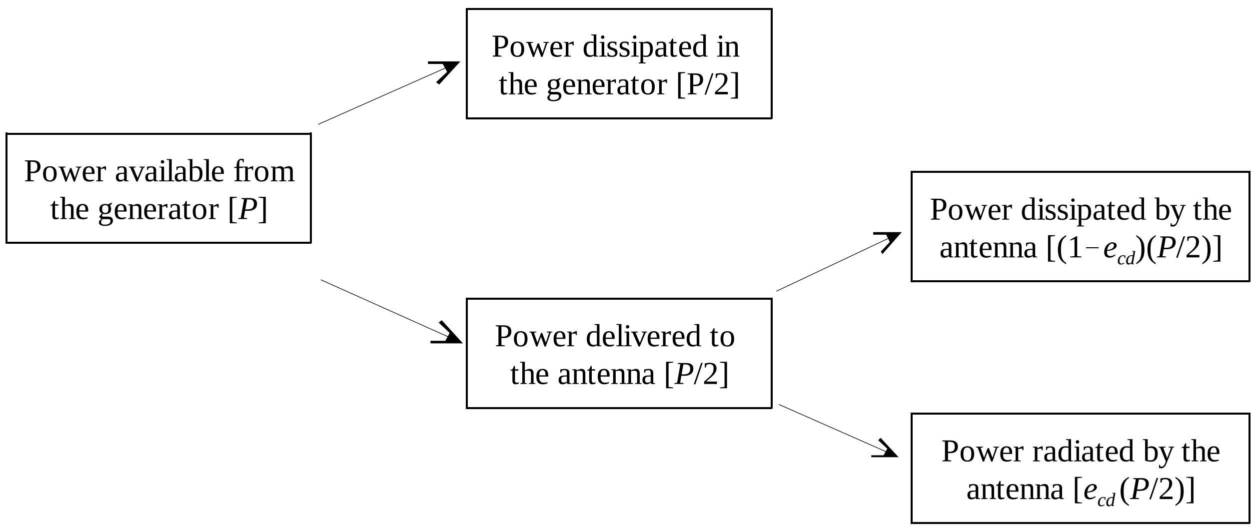
The power available from the generator source is

$$P = \frac{1}{2}V_g I^* = \frac{|V_g|^2}{4(R_r + R_L)}$$

The power dissipated in the generator resistance is

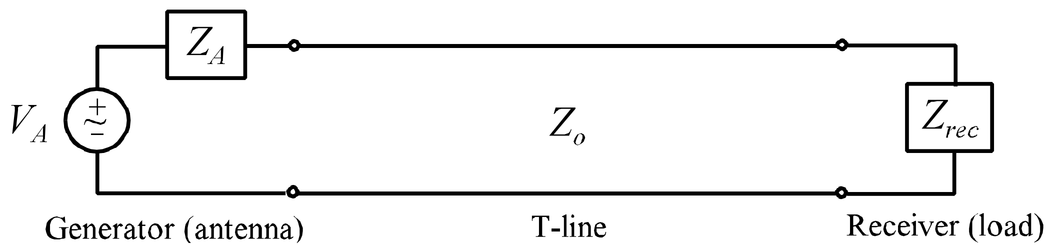
$$P_g = \frac{1}{2}(IR_g)I^* = \frac{1}{2}|I|^2 R_g = \frac{|V_g|^2 R_g}{8(R_r + R_L)^2} = \frac{|V_g|^2}{8(R_r + R_L)} = \frac{1}{2}P$$

<u>Transmitting antenna system summary</u> (maximum power transfer)

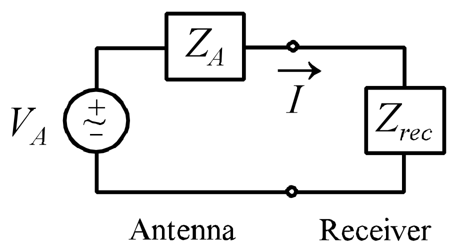


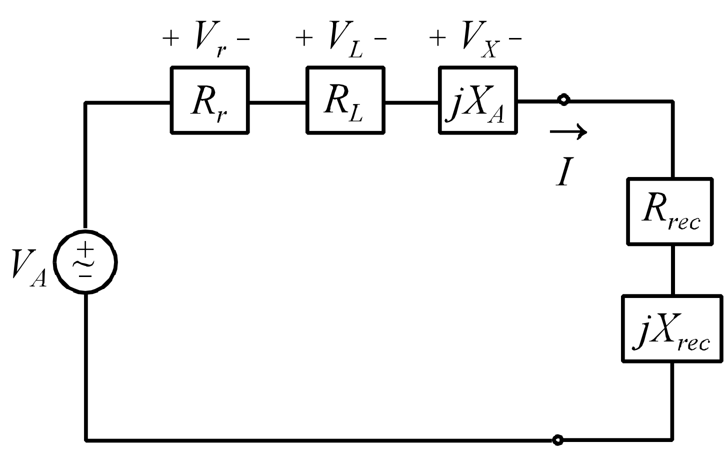
With an ideal transmitting antenna (e_{cd} = 1) given maximum power transfer, one-half of the power available from the generator is radiated by the antenna.

The typical receiving system can be defined by a generator (receiving antenna), transmission line and load (receiver) as shown below.



Assuming the receiving antenna is connected directly to the receiver





For the receiving system, maximum power transfer occurs when

$$Z_A = Z_{rec}^*$$

$$R_A = R_r + R_L = R_{rec}$$

$$X_A = -X_{rec}$$

The circuit current in this case is

$$I = \frac{V_A}{Z_g + Z_A} = \frac{V_A}{2(R_r + R_L)}$$

The power captured by the receiving antenna is

$$P = \frac{1}{2} V_A I^* = \frac{|V_A|^2}{4(R_r + R_L)}$$

Some of the power captured by the receiving antenna is re-radiated (scattered). The power scattered by the antenna (P_{scat}) is

$$P_{scat} = \frac{1}{2} V_r I^* = \frac{|V_A|^2 R_r}{8(R_r + R_L)^2} = e_{cd} \frac{P}{2}$$

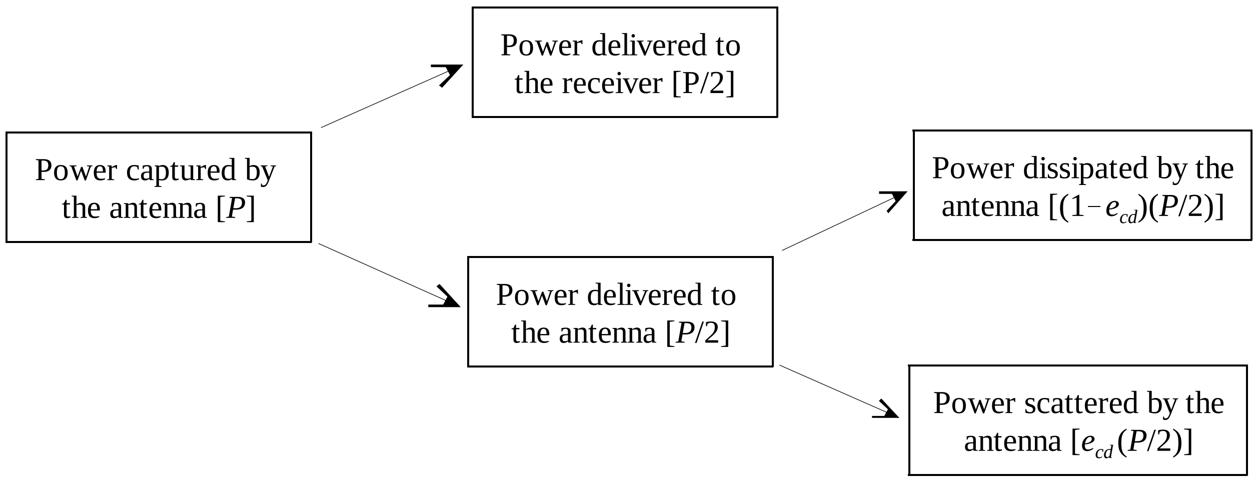
The power dissipated by the receiving antenna in the form of heat is

$$P_L = \frac{1}{2} V_L I^* = \frac{|V_A|^2 R_L}{8(R_r + R_L)^2} = (1 - e_{cd}) \frac{P}{2}$$

The power delivered to the receiver is

$$P_{rec} = \frac{1}{2}(I R_{rec}) I^* = \frac{1}{2} |I|^2 R_{rec} = \frac{|V_A|^2 R_{rec}}{8(R_r + R_L)^2} = \frac{|V_A|^2}{8(R_r + R_L)} = \frac{1}{2} P$$

Receiving antenna system summary (maximum power transfer)



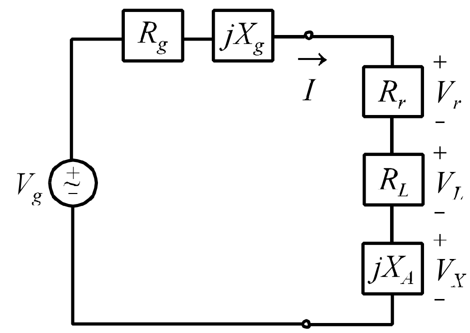
With an ideal receiving antenna (e_{cd} = 1) given maximum power transfer, one-half of the power captured by the antenna is re-radiated (scattered) by the antenna.

Antenna Radiation Efficiency

The radiation efficiency (e_{cd}) of a given antenna has previously been defined in terms of the total power radiated by the antenna (P_{rad}) and the total power dissipated by the antenna in the form of ohmic losses (P_{ohmic}).

$$e_{cd} = \frac{P_{rad}}{P_{in}} = \frac{P_{rad}}{P_{rad} + P_{ohmic}}$$

The total radiated power and the total ohmic losses were determined for the general case of a transmitting antenna using the equivalent circuit. The total radiated power is that "dissipated" in the antenna radiation resistance (R_r).



$$P_{rad} = P_r = \frac{|V_g|^2 R_r}{8(R_r + R_L)^2}$$

The total ohmic losses for the antenna are those dissipated in the antenna loss resistance (R_L).

$$P_{ohmic} = P_L = \frac{|V_g|^2 R_L}{8(R_r + R_L)^2}$$

Inserting the equivalent circuit results for P_{rad} and P_{ohmic} into the equation for the antenna radiation efficiency yields

$$e_{cd} = \; = \frac{R_r}{R_r + R_L}$$

Thus, the antenna radiation efficiency may be found directly from the antenna equivalent circuit parameters.

Antenna Loss Resistance

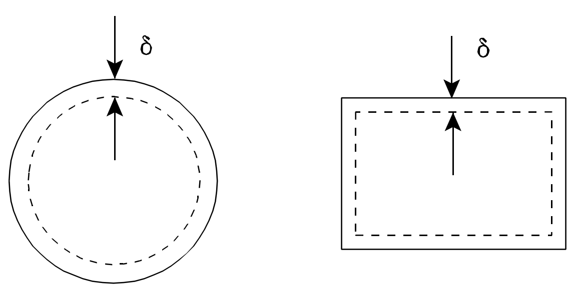
The antenna loss resistance (conductor and dielectric losses) for many antennas is typically difficult to calculate. In these cases, the loss resistance is normally measured experimentally. However, the loss resistance of wire antennas can be calculated easily and accurately. Assuming a conductor of length *l* and cross-sectional area *A* which carries a uniform current density, the DC resistance is

$$R_{DC} = \frac{l}{\sigma A}$$

where σ is the conductivity of the conductor. At high frequencies, the current tends to crowd toward the outer surface of the conductor (skin effect). The HF resistance can be defined in terms of the skin depth δ.

$$\delta = \frac{l}{\sqrt{\pi f \mu \sigma}}$$

where μ is the permeability of the material and *f* is the frequency in Hz.



The skin depth for copper ($\sigma = 5.8\times10^{7}$ ℧/m, $\mu = \mu_o = 4\pi\times10^{-7}$ H/m) may be written as

$$\delta = \frac{66.1}{\sqrt{f}} \text{ (mm)} \qquad \text{[frequency in Hz]}$$

If we define the perimeter distance of the conductor as d_p, then the HF resistance of the conductor can be written as

$$R_{HF} = \frac{l}{\sigma d_p \delta} = \frac{l}{d_p} R_s$$

where R_s is defined as the surface resistance of the material.

$$R_s = \frac{l}{\sigma\delta} = \sqrt{\frac{\pi f \mu}{\sigma}}$$

For the R_{HF} equation to be accurate, the skin depth should be a small fraction of the conductor maximum cross-sectional dimension. In the case of a cylindrical conductor ($d_p \approx 2\pi a$), the HF resistance is

$$R_{HF} = \frac{l}{2\pi a}\sqrt{\frac{\pi f \mu}{\sigma}} = \frac{l}{2a}\sqrt{\frac{f\mu}{\pi\sigma}}$$

f	δ	R
0	∞	R_{DC} = 0.818 mΩ
1 kHz	2.09 mm	~
10 kHz	0.661 mm	R_{HF} = 1.60 mΩ
100 kHz	0.209 mm	R_{HF} = 5.07 mΩ
1 MHz	0.0661 mm	R_{HF} = 16.0 mΩ

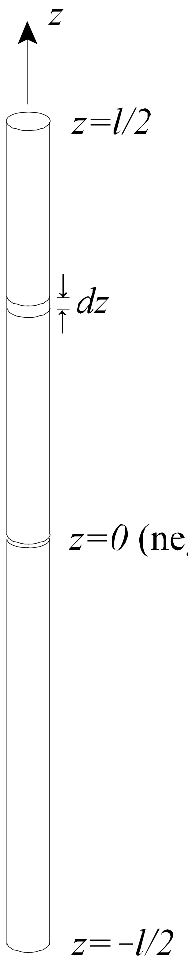
Resistance of 1 m of #10 AWG (a = 2.59 mm) copper wire.

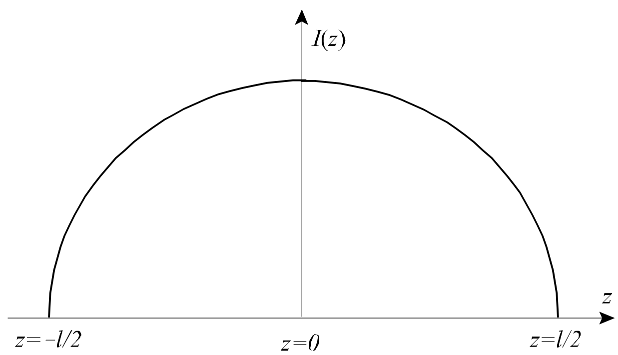
The high frequency resistance formula assumes that the current through the conductor is sinusoidal in time and independent of position along the conductor [$I_z(z,t) = I_o\cos(\omega t)$]. On most antennas, the current is not necessarily independent of position. However, given the actual current distribution on the antenna, an equivalent R_L can be calculated.

Example (Problem 2.44) [Loss resistance calculation]

A dipole antenna consists of a circular wire of length l. Assuming the current distribution on the wire is cosinusoidal, i.e.,

$$I_z(z,t) = I_z(z)\cos(\omega t) = I_o \cos\left(\frac{\pi z'}{l}\right)\cos(\omega t) \qquad -\frac{l}{2} \le z' \le \frac{l}{2}$$





$$P_L = \frac{1}{2} I_o^2 R_L$$ Equivalent circuit equation (uniform current, I_o - peak)

$$P_L = \int dP_L(z)$$ Integration of incremental power along the antenna

$$dP_L(z) = \frac{1}{2}[I(z)]^2 dR_{HF}(z) \qquad\qquad R_{HF} = \frac{l}{d_p} R_s$$

$$= \frac{1}{2}[I(z)]^2 \left[\frac{dz}{d_p} R_s\right]$$

$$= \frac{1}{2} I_o^2 \cos^2\left(\frac{\pi z}{l}\right) \frac{dz}{2\pi a} R_s$$

$$= \frac{I_o^2 R_s}{4\pi a} \cos^2\left(\frac{\pi z}{l}\right) dz$$

$$P_L = \int_{-l/2}^{l/2} dP_L(z)$$

$$= \frac{I_o^2 R_s}{4\pi a} \int_{-l/2}^{l/2} \cos^2\left(\frac{\pi z}{l}\right) dz$$

$$= \frac{I_o^2 R_s}{8\pi a} \left[z + \frac{l}{2\pi} \sin\left(\frac{2\pi z}{l}\right)\right]_{-l/2}^{l/2}$$

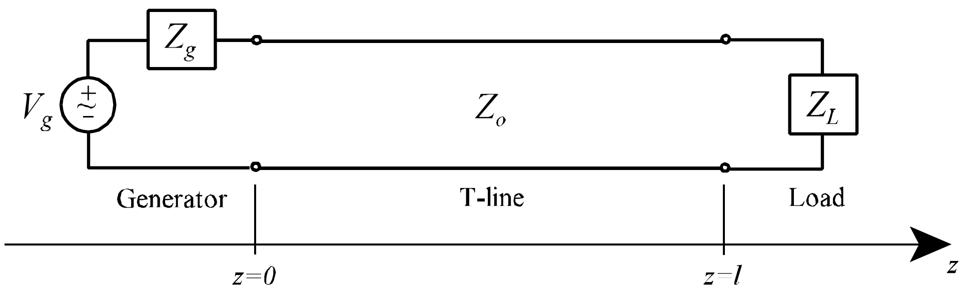
$$= \frac{I_o^2 R_s l}{8\pi a}$$

$$= \frac{1}{2} I_o^2 R_L$$

$$R_L = \frac{1}{2} \frac{l}{2\pi a} R_s = \frac{R_{HF}}{2}$$

Thus, the loss resistance of a dipole antenna of length *l* is one-half that of a the same conductor carrying a uniform current.

Lossless Transmission Line Fundamentals



Transmission line equations (voltage and current)

$$V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}$$

$$I(z) = I_o^+ e^{-j\beta z} + I_o^- e^{j\beta z}$$

~~~~~~~~ ~~~~~~~~

+z directed waves   −z directed waves

V_o^+, V_o^- (voltage coefficients, forward and reverse waves)

I_o^+, I_o^- (current coefficients, forward and reverse waves)

$$\beta = \frac{\omega}{u} \quad \text{(phase constant)} \qquad u = \frac{1}{\sqrt{\mu\epsilon}} \quad \text{(wave velocity)}$$

$$\frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = Z_o \quad \text{(characeristic impedance)}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad \text{(lossy line)}$$

$$= \sqrt{\frac{L}{C}} \quad \text{(lossless line)}$$
~~~~~~~~

$$V(z) = V_o^+ e^{-j\beta z}\left[1 + \frac{V_o^-}{V_o^+} e^{j2\beta z}\right] = V_o^+ e^{-j\beta z}\left[1 + \Gamma(z)\right]$$

$$I(z) = \frac{V_o^+}{Z_o} e^{-j\beta z}\left[1 - \frac{V_o^-}{V_o^+} e^{j2\beta z}\right] = \frac{V_o^+}{Z_o} e^{-j\beta z}\left[1 - \Gamma(z)\right]$$

$$\Gamma(z) = \frac{V_o^-}{V_o^+} e^{j2\beta z}$$ (reflection coefficient definition)

$$\Gamma(z) = \frac{Z_L - Z_o}{Z_L + Z_o} e^{j2\beta(z - l)}$$ (reflection coefficient at any point on a terminated t−line)

$$\Gamma(l) = \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o}$$ (reflection coefficient at the load of a terminated t−line)

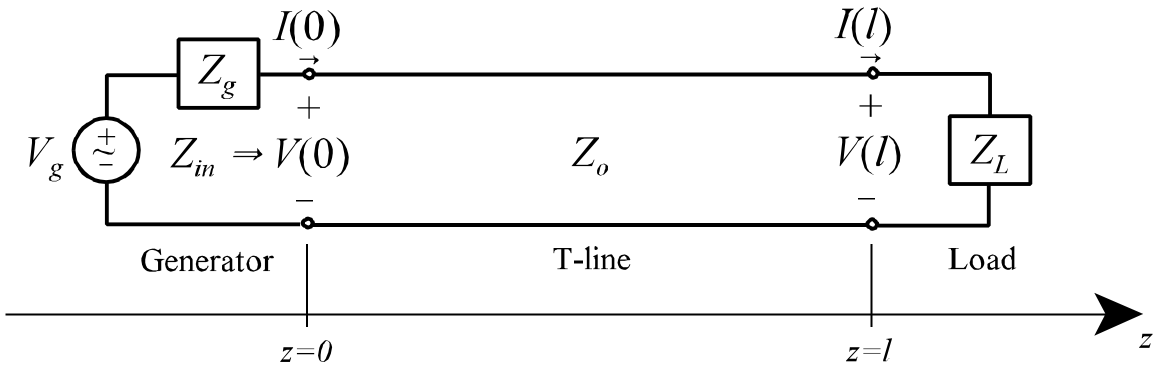
$$Z_{in}(z) = \frac{V(z)}{I(z)} = Z_o \frac{Z_L + jZ_o \tan[\beta(l - z)]}{Z_o + jZ_L \tan[\beta(l - z)]}$$ (input impedance at any point)

$$Z_{in}(0) = \frac{V(0)}{I(0)} = Z_o \frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)}$$ (input impedance at t−line input)

$$s = \frac{V_{\max}}{V_{\min}} = \frac{I_{\max}}{I_{\min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$$ (standing wave ratio)

$$|\Gamma_L| = \frac{s - 1}{s + 1}$$ (|reflection coefficient| at the load)

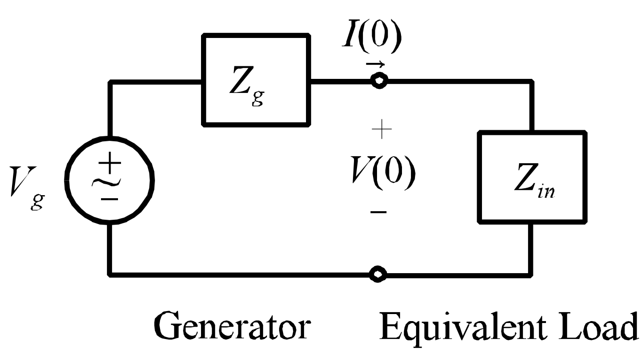
Transmitting/Receiving Systems with Transmission Lines



Using transmission line theory, the impedance seen looking into the input terminals of the transmission line (Z_{in}) is

$$Z_{in}(0) = \frac{V(0)}{I(0)} = Z_o \frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)}$$

The resulting equivalent circuit is shown below.



The current and voltage at the transmission line input terminals are

$$I(0) = \frac{V_g}{Z_g + Z_{in}}$$

$$V(0) = I(0)Z_{in} = V_g \frac{Z_{in}}{Z_g + Z_{in}}$$

The power available from the generator is

$$P_g = \frac{1}{2} V_g I^*(0) = \frac{|V_g|^2}{2(Z_g + Z_{in})^*}$$

The power delivered to the transmission line input terminals is

$$P(0) = \frac{1}{2} V(0) I^*(0) = \frac{Z_{in}}{2} \left| \frac{V_g}{Z_g + Z_{in}} \right|^2$$

The power associated with the generator impedance is

$$P_{Z_g} = \frac{1}{2} [V_g - V(0)] I^*(0) = \frac{Z_g}{2} \left| \frac{V_g}{Z_g + Z_{in}} \right|^2$$

Given the current and the voltage at the input to the transmission line, the values at any point on the line can be found using the transmission line equations.

$$V(z) = V_o^+ e^{-j\beta z} \left[1 + \frac{V_o^-}{V_o^+} e^{j2\beta z} \right] = V_o^+ e^{-j\beta z} \left[1 + \Gamma(z) \right]$$

$$I(z) = \frac{V_o^+}{Z_o} e^{-j\beta z} \left[1 - \frac{V_o^-}{V_o^+} e^{j2\beta z} \right] = \frac{V_o^+}{Z_o} e^{-j\beta z} \left[1 - \Gamma(z) \right]$$

The unknown coefficient V_o^+ may be determined from either $V(0)$ or $I(0)$ which were found in the input equivalent circuit. Using $V(0)$ gives

$$V(0) = V_g \frac{Z_{in}}{Z_g + Z_{in}} = V_o^+ [1 + \Gamma(0)]$$

where

$$\Gamma(z) = \frac{Z_L - Z_o}{Z_L + Z_o} e^{j2\beta(z-l)} \qquad \Rightarrow \qquad \Gamma(0) = \frac{Z_L - Z_o}{Z_L + Z_o} e^{-j2\beta l}$$

$$V_o^+ = \frac{V(0)}{1 + \Gamma(0)} = \frac{V_g \dfrac{Z_{in}}{Z_g + Z_{in}}}{\left[1 + \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{-j2\beta l}\right]}$$

Given the coefficient V_o^+, the current and voltage at the load, from the transmission line equations are

$$V(l) = V_o^+ e^{-j\beta l}\left[1 + \Gamma(l)\right] = V_o^+ e^{-j\beta l}\left[1 + \frac{Z_L - Z_o}{Z_L + Z_o}\right]$$

$$I(l) = \frac{V_o^+}{Z_o} e^{-j\beta l}\left[1 - \Gamma(l)\right] = \frac{V_o^+}{Z_o} e^{-j\beta l}\left[1 - \frac{Z_L - Z_o}{Z_L + Z_o}\right]$$

The power delivered to the load is then

$$P(l) = \frac{1}{2} V(l) I^*(l)$$

The complexity of the previous equations leads to solutions which are typically determined by computer or Smith chart.

MATLAB m-file (generator/t-line/load)

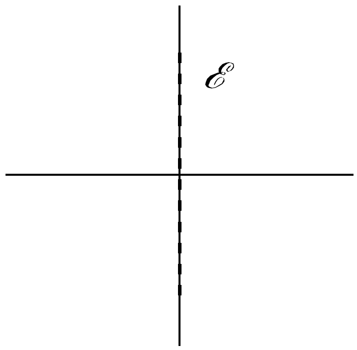
```
Vg=input('Enter the complex generator voltage                 ');
Zg=input('Enter the complex generator impedance               ');
Zo=input('Enter the lossless t-line characteristic impedance ');
l=input('Enter the lossless t-line length in wavelengths    ');
Zl=input('Enter the complex load impedance                    ');
j=0+1j;
betal=2*pi*l;
Zin=Zo*(Zl+j*Zo*tan(betal))/(Zo+j*Zl*tan(betal));
gammal=(Zl-Zo)/(Zl+Zo);
gamma0=gammal*exp(-j*2*betal);
Ig=Vg/(Zg+Zin);
Pg=0.5*Vg*conj(Ig);
V0=Ig*Zin;
P0=0.5*V0*conj(Ig);
Vcoeff=V0/(1+gamma0);
Vl=Vcoeff*exp(-j*betal)*(1+gammal);
Il=Vcoeff*exp(-j*betal)*(1-gammal)/Zo;
Pl=0.5*Vl*conj(Il);
s=(1+abs(gammal))/(1-abs(gammal));
format compact
Generator_voltage=Vg
Generator_current=Ig
Generator_power=Pg
Generator_impedance_voltage=Vg-V0
Generator_impedance_current=Ig
Generator_impedance_power=Pg-P0
T_line_input_voltage=V0
T_line_input_current=Ig
T_line_input_power=P0
T_line_input_impedance=Zin
T_line_input_reflection_coeff=gamma0
T_line_standing_wave_ratio=s
Load_voltage=Vl
Load_current=Il
Load_power=Pl
Load_reflection_coeff=gammal
```

Given $V_g = (10+j0)$ V, $Z_g = (100+j0)\ \Omega$ and $l = 5.125\lambda$, the following results are found.

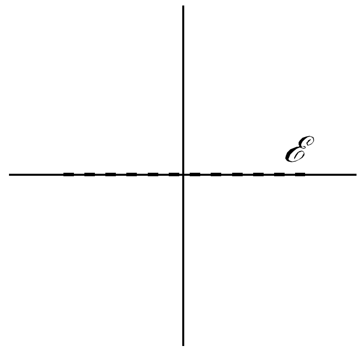
Z_o	Z_L	Z_{in}	$\lvert\Gamma(0)\rvert = \lvert\Gamma(l)\rvert$	P_g	s	$P(l)$
100	75	96+j28	0.1429	0.25	1.3333	0.1224
100	100	100	0	0.25	1	0.125
100	125	98−j22	0.1111	0.25	1.25	0.1235
75	100	72−j21	0.1429	0.2864	1.3333	0.1199
100	100	100	0	0.25	1	0.125
125	100	122+j27	0.1111	0.2219	1.25	0.1219

Antenna Polarization

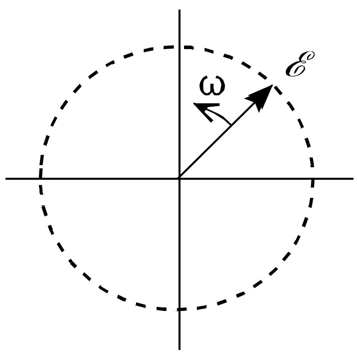
The polarization of an plane wave is defined by the figure traced by the instantaneous electric field at a fixed observation point. The following are the most commonly encountered polarizations assuming the wave is approaching.



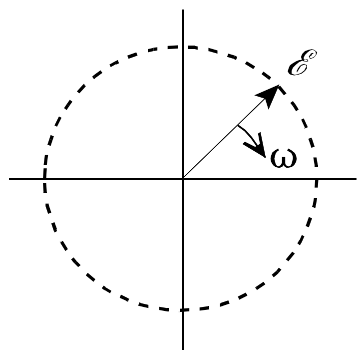
Vertical linear



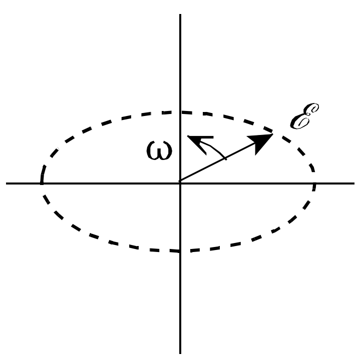
Horizontal linear



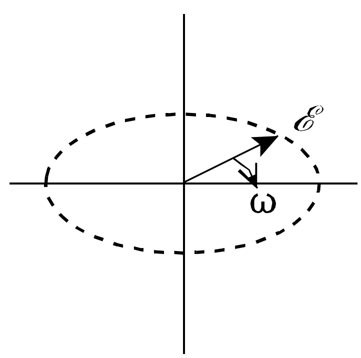
Right-hand circular



Left-hand circular



Right-hand elliptical



Left-hand elliptical

The polarization of the antenna in a given direction is defined as the polarization of the wave radiated in that direction by the antenna. Note that any of the previous polarization figures may be rotated by some arbitrary angle.

<u>Polarization loss factor</u>

Incident wave polarization

$$\boldsymbol{E_i} = \boldsymbol{a_i} E_i$$

Antenna polarization

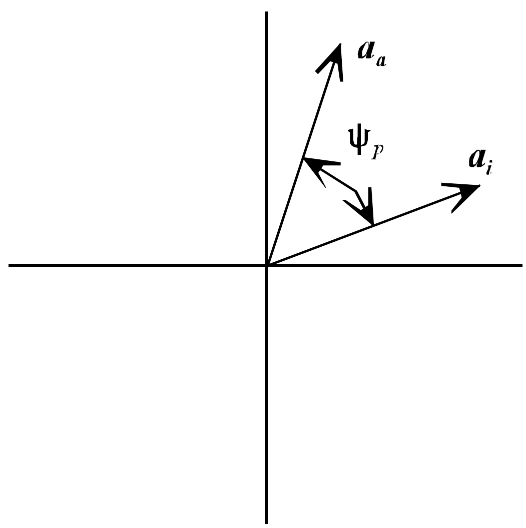
$$\boldsymbol{E_a} = \boldsymbol{a_a} E_a$$

Polarization loss factor (*PLF*)

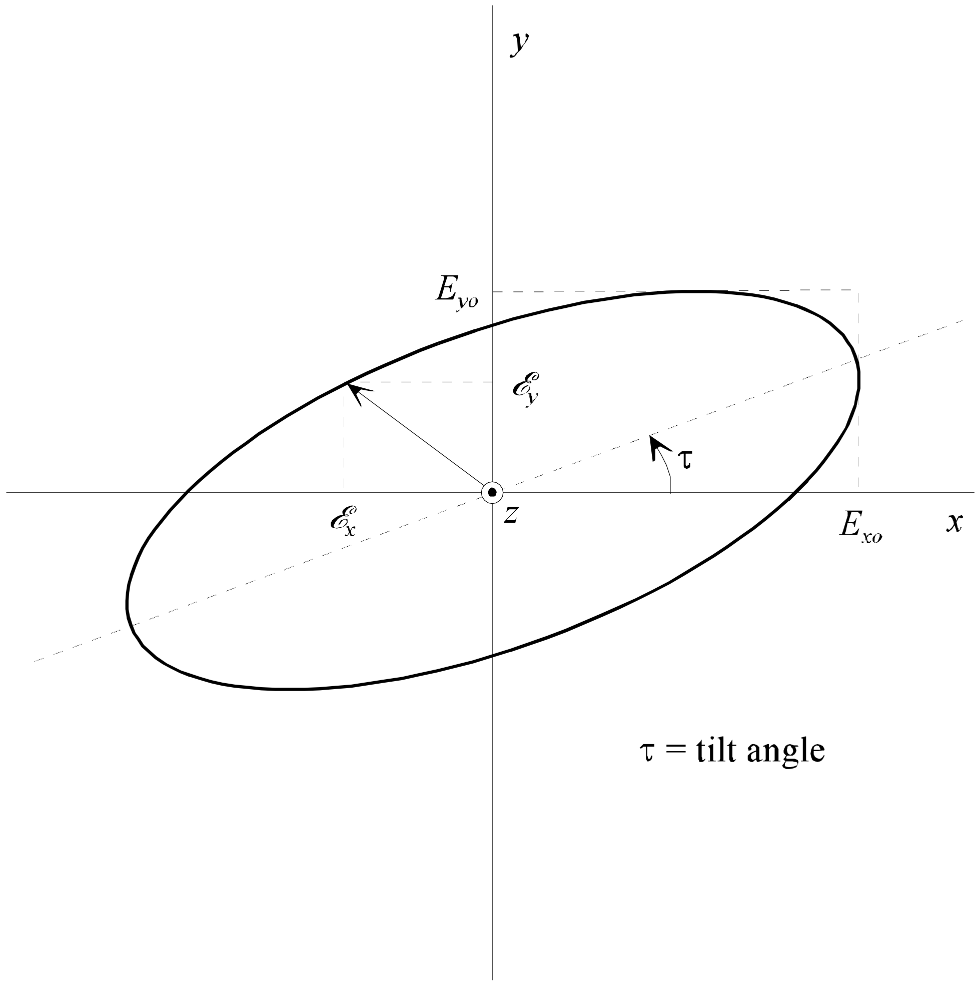
$$PLF = |\boldsymbol{a_i} \cdot \boldsymbol{a_a}|^2 = |\cos\psi_p|^2$$

PLF in dB

$$PLF(dB) = 10\log_{10}(PLF)$$



General Polarization Ellipse



The vector electric field associated with a +*z*-directed plane wave can be written in general phasor form as

$$\boldsymbol{E} = (E_x \boldsymbol{a_x} + E_y \boldsymbol{a_y})\, e^{-jkz}$$

where E_x and E_y are complex phasors which may be defined in terms of magnitude and phase.

$$E_x = E_{xo} e^{j\phi_x} \qquad\qquad E_y = E_{yo} e^{j\phi_y}$$

The instantaneous components of the electric field are found by multiplying the phasor components by $e^{j\omega t}$ and taking the real part.

$$\mathscr{E}_x(z,t) = \mathrm{Re}\,[\,E_{xo}\, e^{j\phi_x} e^{-jkz}\, e^{j\omega t}\,] = E_{xo}\cos(\omega t - kz + \phi_x)$$

$$\mathscr{E}_y(z,t) = \mathrm{Re}\,[\,E_{yo}\, e^{j\phi_y} e^{-jkz}\, e^{j\omega t}\,] = E_{yo}\cos(\omega t - kz + \phi_y)$$

The relative positions of the instantaneous electric field components on the general polarization ellipse defines the polarization of the plane wave.

<u>Linear Polarization</u>

If we define the phase shift between the two electric field components as

$$\Delta\phi = \phi_y - \phi_x$$

we find that a phase shift of

$$\Delta\phi = \phi_y - \phi_x = n\pi \qquad n = 0, 1, 2, \ldots$$

defines a linearly polarized wave.

$$\mathscr{E}_x(z,t) = E_{xo}\cos(\omega t - kz + \phi_x)$$

$$\mathscr{E}_y(z,t) = E_{yo}\cos(\omega t - kz + \phi_x + n\pi) = \pm E_{yo}\cos(\omega t - kz + \phi_x) \quad \begin{Bmatrix} n \;\; even \\ n \;\; odd \end{Bmatrix}$$

Examples of linear polarization:

If $E_{yo} = 0$ $\Rightarrow$ Linear polarization in the x-direction ($\tau = 0$)
If $E_{xo} = 0$ $\Rightarrow$ Linear polarization in the y-direction ($\tau = 90^o$)
If $E_{xo} = E_{yo}$ and n is even $\Rightarrow$ Linear polarization ($\tau = 45^o$)
If $E_{xo} = E_{yo}$ and n is odd $\Rightarrow$ Linear polarization ($\tau = 135^o$)

Circular Polarization

If $E_{xo} = E_{yo}$ and

$$\Delta\phi = \phi_y - \phi_x = (2n + \tfrac{1}{2})\pi \qquad n = 0, 1, 2, \ldots$$

then

$$\mathscr{E}_x(z,t) = E_{xo}\cos(\omega t - kz + \phi_x)$$

$$\mathscr{E}_y(z,t) = E_{xo}\cos[\omega t - kz + \phi_x + (2n + \tfrac{1}{2})\pi] = -E_{xo}\sin(\omega t - kz + \phi_x)$$

This is left-hand circular polarization.

If $E_{xo} = E_{yo}$ and

$$\Delta\phi = \phi_y - \phi_x = -(2n + \tfrac{1}{2})\pi \qquad n = 0, 1, 2, \ldots$$

then

$$\mathscr{E}_x(z,t) = E_{xo}\cos(\omega t - kz + \phi_x)$$

$$\mathscr{E}_y(z,t) = E_{xo}\cos[\omega t - kz + \phi_x - (2n + \tfrac{1}{2})\pi] = E_{xo}\sin(\omega t - kz + \phi_x)$$

This is right-hand circular polarization.

Elliptical Polarization

Elliptical polarization follows definitions as circular polarization except that $E_{xo} \neq E_{yo}$.

$E_{xo} \neq E_{yo}$, $\Delta\phi = (2n+\tfrac{1}{2})\pi$ $\Rightarrow$ left-hand elliptical polarization
$E_{xo} \neq E_{yo}$, $\Delta\phi = -(2n+\tfrac{1}{2})\pi$ $\Rightarrow$ right-hand elliptical polarization

Antenna Equivalent Areas

Antenna Effective Aperture (Area)

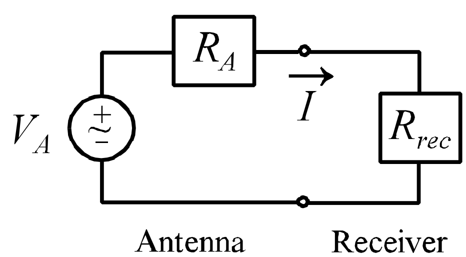
Given a receiving antenna oriented for maximum response, polarization matched to the incident wave, and impedance matched to its load, the resulting power delivered to the receiver (P_{rec}) may be defined in terms of the *antenna effective aperture* (A_e) as

$$P_{rec} = SA_e \qquad \text{(W)}$$

where S is the power density of the incident wave (magnitude of the Poynting vector) defined by

$$S = \frac{1}{2}|\boldsymbol{E_i} \times \boldsymbol{H_i^*}| \qquad \text{(W/m}^2\text{)}$$

According to the equivalent circuit under matched conditions,



$$R_A = R_r + R_L = R_{rec}$$

$$P_{rec} = \frac{1}{2}V_{rec}I^* = \frac{1}{2}\frac{V_A}{2}\left(\frac{V_A}{2R_A}\right)^* = \frac{|V_A|^2}{8R_A}$$

We may solve for the antenna effective aperture which gives

$$A_e = \frac{P_{rec}}{S} = \frac{|V_A|^2}{8SR_A} = \frac{|V_A|^2}{8S(R_r + R_L)}$$

Antenna Scattering Area

The total power scattered by the receiving antenna is defined as the product of the incident power density and the *antenna scattering area* (A_s).

$$P_s = S A_s$$

From the equivalent circuit, the total scattered power is

$$P_s = \frac{|V_A|^2 R_r}{8(R_r + R_L)^2}$$

which gives

$$A_s = \frac{P_s}{S} = \frac{|V_A|^2 R_r}{8S(R_r + R_L)^2}$$

Antenna Loss Area

The total power dissipated as heat by the receiving antenna is defined as the product of the incident power density and the *antenna loss area* (A_L).

$$P_L = S A_L$$

From the equivalent circuit, the total dissipated power is

$$P_L = \frac{|V_A|^2 R_L}{8(R_r + R_L)^2}$$

which gives

$$A_L = \frac{P_L}{S} = \frac{|V_A|^2 R_L}{8S(R_r + R_L)^2}$$

Antenna Capture Area

The total power captured by the receiving antenna (power delivered to the load + power scattered by the antenna + power dissipated in the form of heat) is defined as the product of the incident power density and the *antenna capture area* (A_c).

$$P_c = SA_c$$

The total power captured by the antenna is

$$P_c = P_{rec} + P_s + P_L = \frac{|V_A|^2}{4(R_r + R_L)}$$

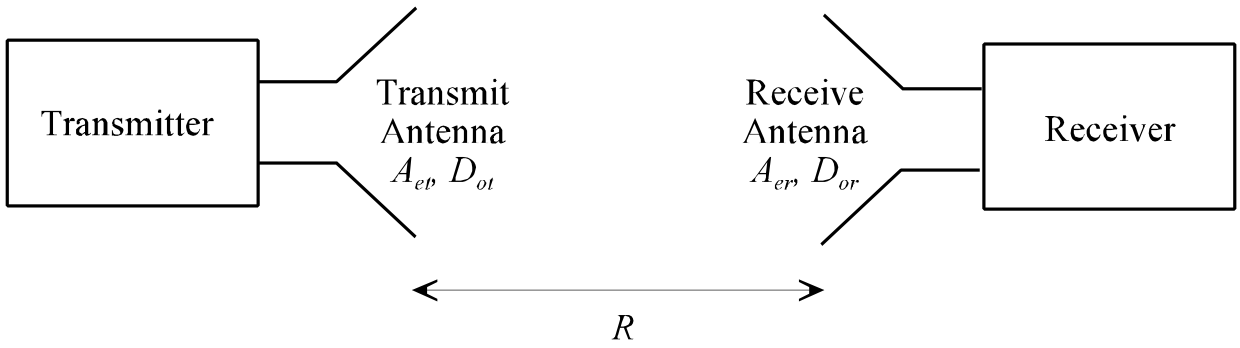
which gives

$$A_c = \frac{P_c}{S} = \frac{|V_A|^2}{4S(R_r + R_L)}$$

Note that $A_c = A_e + A_s + A_L$.

Maximum Directivity and Effective Aperture

Assume the transmitting and receiving antennas are lossless and oriented for maximum response.



A_{et}, D_{ot} - transmit antenna effective aperture and maximum directivity
A_{er}, D_{or} - receive antenna effective aperture and maximum directivity

If we assume that the total power transmitted by the transmit antenna is P_t, the power density at the receive antenna (W_r) is

$$W_r = \frac{P_t}{4\pi R^2} D_{ot}$$

The total power received by the receive antenna (P_r) is

$$P_r = W_r A_{er} = \frac{P_t D_{ot} A_{er}}{4\pi R^2}$$

which gives

$$D_{ot} A_{er} = \frac{P_r}{P_t} 4\pi R^2$$

If we interchange the transmit and receive antennas, the previous equation still holds true by interchanging the respective transmit and receive quantities (assuming a linear, isotropic medium), which gives

$$D_{or} A_{et} = \frac{P_r}{P_t} 4\pi R^2$$

These two equations yield

$$D_{ot} A_{er} = D_{or} A_{et}$$

or

$$\frac{D_{ot}}{A_{et}} = \frac{D_{or}}{A_{er}}$$

If the transmit antenna is an isotropic radiator, we will later show that

$$D_{ot} = 1 \qquad\qquad A_{et} = \frac{\lambda^2}{4\pi}$$

which gives

$$\frac{D_{or}}{A_{er}} = \frac{4\pi}{\lambda^2} \qquad\qquad (\textbf{for any antenna})$$

Therefore, the equivalent aperture of a lossless antenna may be defined in terms of the maximum directivity as

$$A_e = \left(\frac{\lambda^2}{4\pi} \right) D_o$$

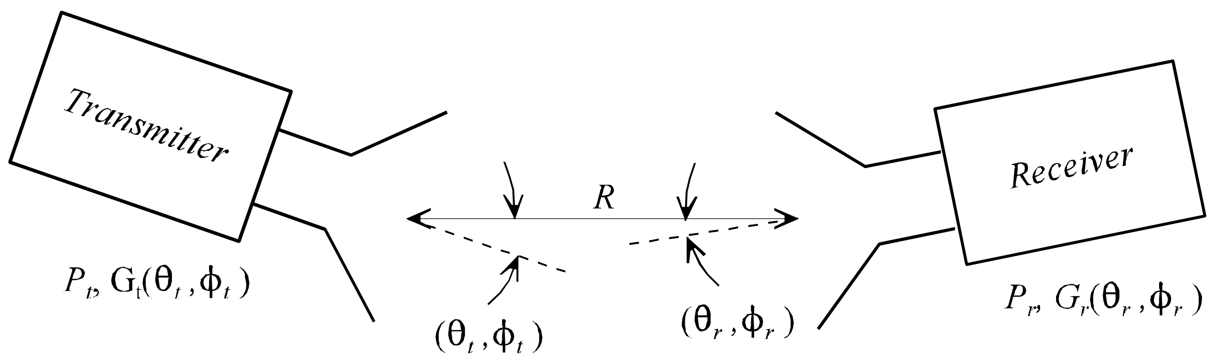
The overall antenna efficiency (e_o) may be included to account for the ohmic losses and mismatch losses in an antenna with losses.

$$A_e = e_o \left(\frac{\lambda^2}{4\pi} \right) D_o = e_{cd} (1 - |\Gamma|^2) \left(\frac{\lambda^2}{4\pi} \right) D_o$$

The effect of polarization loss can also be included to yield

$$A_e = e_{cd} (1 - |\Gamma|^2) \left(\frac{\lambda^2}{4\pi} \right) D_o \, |\boldsymbol{a}_t \cdot \boldsymbol{a}_r|^2$$

Friis Transmission Equation



The *Friis transmission equation* defines the relationship between transmitted power and received power in an arbitrary transmit/receive antenna *system*. Given arbitrarily oriented transmitting and receiving antennas, the power density at the receiving antenna (W_r) is

$$W_r = \frac{P_t}{4\pi R^2} G_t(\theta_t, \phi_t) = e_{ot} \frac{P_t}{4\pi R^2} D_t(\theta_t, \phi_t)$$

where P_t is the input power at the terminals of the transmit antenna and where the transmit antenna gain and directivity for the system performance are related by the overall efficiency

$$G_t(\theta_t, \phi_t) = e_{ot} D_t(\theta_t, \phi_t) = e_{cdt}(1 - |\Gamma_t|^2) D_t(\theta_t, \phi_t)$$

where e_{cdt} is the radiation efficiency of the transmit antenna and Γ_t is the reflection coefficient at the transmit antenna terminals. Notice that this definition of the transmit antenna gain includes the mismatch losses for the transmit system in addition to the conduction and dielectric losses. A manufacturer's specification for the antenna gain will not include the mismatch losses.

The total received power delivered to the terminals of the receiving antenna (P_r) is

$$P_r = W_r A_{er}$$

where the effective aperture of the receiving antenna (A_{er}) must take into account the orientation of the antenna. We may extend our previous definition of the antenna effective aperture (obtained using the maximum directivity) to a general effective aperture for any antenna orientation.

$$A_{er} = e_{or}\left(\frac{\lambda^2}{4\pi}\right) D_{or} \qquad \textbf{(oriented for max response)}$$

$$A_{er}(\theta_r,\phi_r) = e_{or}\left(\frac{\lambda^2}{4\pi}\right) D_r(\theta_r,\phi_r) \qquad \textbf{(arbitrarily oriented)}$$

The total received power is then

$$P_r = e_{or}e_{ot}\left(\frac{\lambda}{4\pi R}\right)^2 D_r(\theta_r,\phi_r)D_t(\theta_t,\phi_t)\,P_t$$

such that the ratio of received power to transmitted power is

$$\frac{P_r}{P_t} = e_{cdr}\,e_{cdt}(1-|\Gamma_t|^2)(1-|\Gamma_r|^2)\left(\frac{\lambda}{4\pi R}\right)^2 D_r(\theta_r,\phi_r)D_t(\theta_t,\phi_t)$$

Including the polarization losses yields

$$\frac{P_r}{P_t} = e_{cdr}\,e_{cdt}\,(1-|\Gamma_t|^2)(1-|\Gamma_r|^2)$$
$$\times\left(\frac{\lambda}{4\pi R}\right)^2 D_r(\theta_r,\phi_r)D_t(\theta_t,\phi_t)\,|\boldsymbol{a}_t\cdot\boldsymbol{a}_r|^2$$

(General Friis transmission formula)

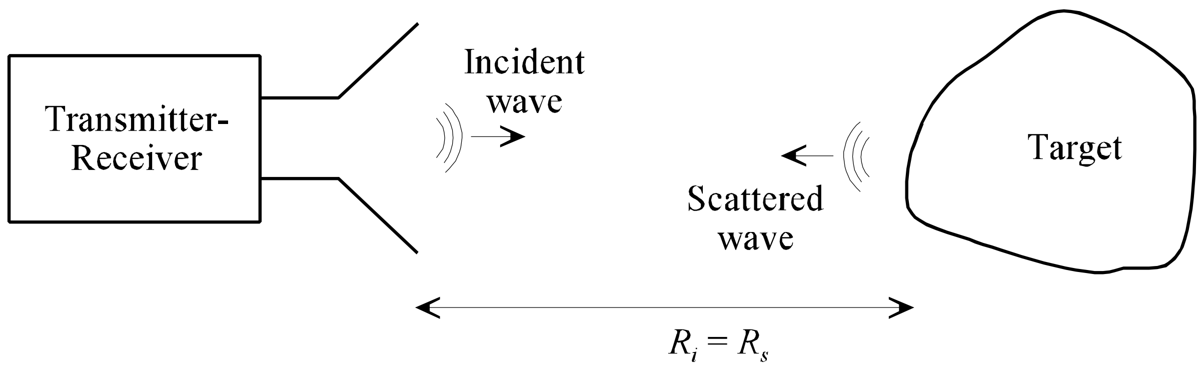
For antennas aligned for maximum response, reflection-matched and polarization matched, the Friis transmission equation reduces to

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 G_{ot}G_{or}$$

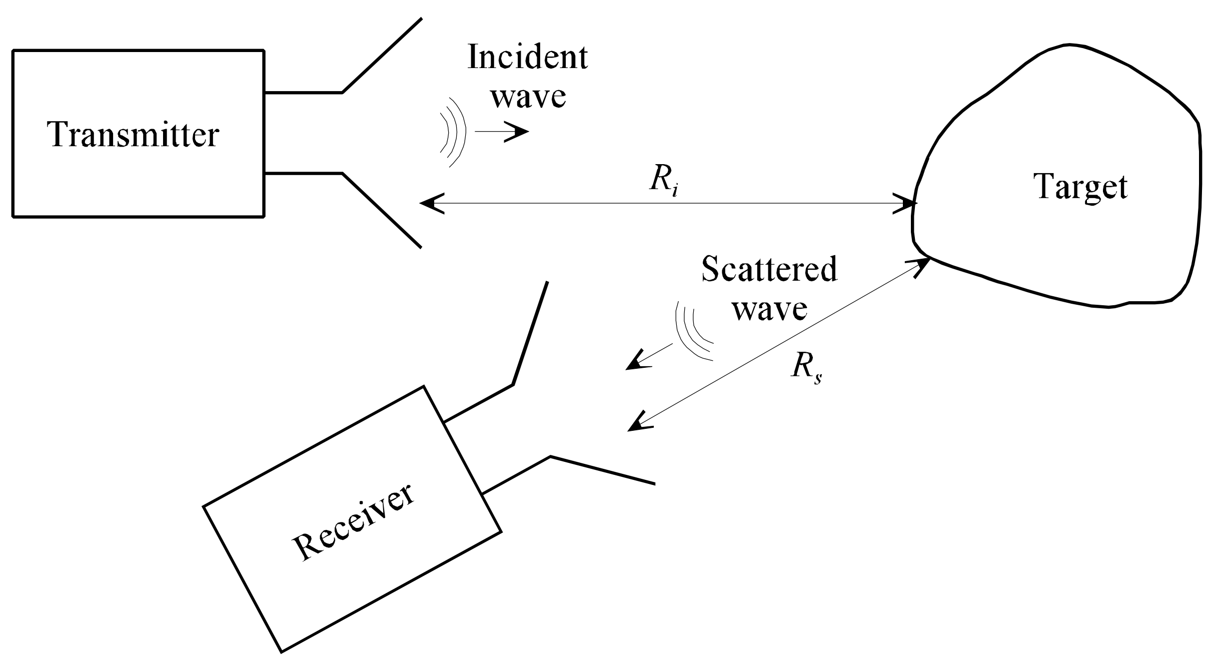
Radar Range Equation and Radar Cross Section

The Friis transmission formula can be used to determine the radar range equation. In order to determine the maximum range at which a given target can be detected by radar, the type of radar system (monostatic or bistatic) and the scattering properties of the target (radar cross section) must be known.

Monostatic radar system - transmit and receive antennas at the same location.



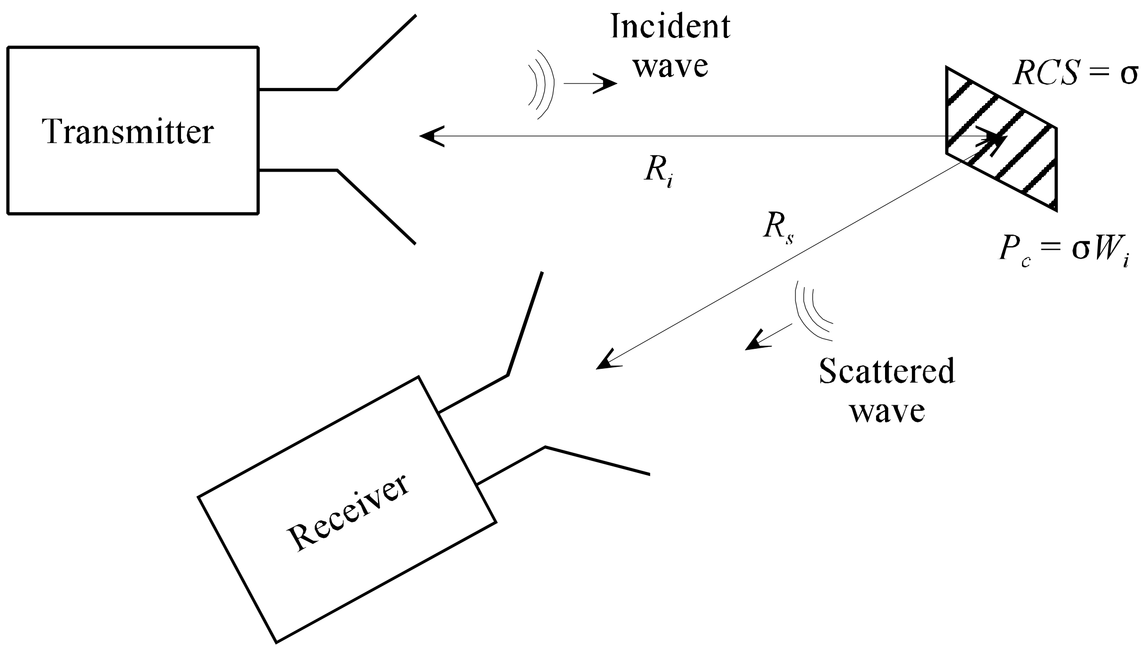
Bistatic radar system- transmit and receive antennas at separate locations.



Radar cross section (RCS) - a measure of the ability of a target to reflect (scatter) electromagnetic energy (units = m^2). The area which intercepts that amount of total power which, when scattered isotropically, produces the same power density at the receiver as the actual target.

If we define

σ = radar cross section (m^2)
W_i = incident power density at the target (W/m^2)
P_c = equivalent power captured by the target (W)
W_s = scattered power density at the receiver (W/m^2)



According to the definition of the target RCS, the relationship between the incident power density at the target and the scattered power density at the receive antenna is

$$\lim_{R_s \to \infty} \left[\frac{\sigma W_i}{4 \pi R_s^2} \right] = W_s$$

The limit is usually included since we must be in the far-field of the target for the radar cross section to yield an accurate result.

The radar cross section may be written as

$$\sigma = \lim_{R_s \to \infty} \left[4\pi R_s^2 \frac{W_s}{W_i} \right] = \lim_{R_s \to \infty} \left[4\pi R_s^2 \frac{|\boldsymbol{E}_s^2|}{|\boldsymbol{E}_i^2|} \right] = \lim_{R_s \to \infty} \left[4\pi R_s^2 \frac{|\boldsymbol{H}_s^2|}{|\boldsymbol{H}_i^2|} \right]$$

where ($\boldsymbol{E}_i$, $\boldsymbol{H}_i$) are the incident electric and magnetic fields at the target and ($\boldsymbol{E}_s$, $\boldsymbol{H}_s$) are the scattered electric and magnetic fields at the receiver. The incident power density at the target generated by the transmitting antenna (P_t, G_t, D_t, e_{ot}, Γ_t, $\boldsymbol{a}_t$) is given by

$$W_i = \frac{P_t}{4\pi R_i^2} G_t(\theta_t, \phi_t)$$

The total power captured by the target (P_c) is

$$P_c = \sigma W_i = \frac{\sigma P_t}{4\pi R_i^2} G_t(\theta_t, \phi_t) = \frac{e_{ot} \sigma P_t}{4\pi R_i^2} D_t(\theta_t, \phi_t)$$

The power captured by the target is scattered isotropically so that the scattered power density at the receiver is

$$W_s = \frac{P_c}{4\pi R_s^2} = \frac{e_{ot} \sigma P_t}{(4\pi R_i R_s)^2} D_t(\theta_t, \phi_t)$$

The power delivered to the receiving antenna load is

$$P_r = W_s A_{er} = W_s\, e_{or} D_t(\theta_t, \phi_t) \frac{\lambda^2}{4\pi}$$

$$= e_{ot}\, e_{or}\, \sigma \frac{D_t(\theta_t, \phi_t) D_r(\theta_r, \phi_r)}{4\pi} \left(\frac{\lambda}{4\pi R_i R_s} \right)^2 P_t$$

Showing the conduction losses, mismatch losses and polarization losses explicitly, the ratio of the received power to transmitted power becomes

$$\frac{P_r}{P_t} = e_{cdt}e_{cdr}\left(1 - |\boldsymbol{\Gamma}_t|^2\right)\left(1 - |\boldsymbol{\Gamma}_r|^2\right)$$

$$\times\, \sigma\, \frac{D_t(\theta_t,\phi_t)\, D_r(\theta_r,\phi_r)}{4\pi}\left(\frac{\lambda}{4\pi R_i R_s}\right)^2 |\boldsymbol{a}_w \cdot \boldsymbol{a}_r|^2$$

(radar range equation)

where

$\boldsymbol{a}_w$ - polarization unit vector for the scattered waves
$\boldsymbol{a}_r$ - polarization unit vector for the receive antenna

Given matched antennas aligned for maximum response and polarization matched, the general radar range equation reduces to

$$\frac{P_r}{P_t} = \sigma\, \frac{G_{ot}G_{or}}{4\pi}\left(\frac{\lambda}{4\pi R_i R_s}\right)^2$$

Example

Problem 2.65 A radar antenna, used for both transmitting and receiving, has a gain of 150 at its operating frequency of 5 GHz. It transmits 100 kW, and is aligned for maximum directional radiation and reception to a target 1 km away having a cross section of 3 m^2. The received signal matches the polarization of the transmitted signal. Find the received power.

$$P_r = P_t \sigma \frac{G_{ot} G_{or}}{4\pi} \left(\frac{\lambda}{4\pi R_i R_s} \right)^2$$

$$P_t = 10^5 \ W \qquad R_i = R_s = 10^3 \ m \qquad G_{ot} = G_{or} = 150$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8 \ m/s}{5\times10^9 \ Hz} = 0.06 \ m$$

$$P_r = (10^5)(3) \frac{150^2}{4\pi} \left(\frac{0.06}{4\pi\, 10^6} \right)^2 = 12.2 \ nW$$

$$W_i = \frac{P_t}{4\pi R_i^2} G_t = \frac{10^5}{4\pi\, 10^6} 150 = 1.19 \ W/m^2$$

$$P_c = \sigma W_i = (3)(1.19) = 3.58 \ W$$

$$W_s = \frac{P_c}{4\pi R_s^2} = \frac{3.58}{4\pi\, 10^6} = 285 \ nW/m^2$$

$$P_r = W_s A_{er} = W_s \left(\frac{\lambda^2}{4\pi} \right) G_{or} = 285\times10^{-9} \frac{0.06^2}{4\pi} 150 = 12.2 \ nW$$